

## Multidimensional Gaussian sums arising from the distribution of Birkhoff sums in zero entropy dynamical systems

This article has been downloaded from IOPscience. Please scroll down to see the full text article.

2004 J. Phys. A: Math. Gen. 37 11759

(<http://iopscience.iop.org/0305-4470/37/49/003>)

View [the table of contents for this issue](#), or go to the [journal homepage](#) for more

Download details:

IP Address: 171.66.16.65

The article was downloaded on 02/06/2010 at 19:46

Please note that [terms and conditions apply](#).

# Multidimensional Gaussian sums arising from the distribution of Birkhoff sums in zero entropy dynamical systems

M Bernardo<sup>1,2</sup>, M Courbage<sup>1</sup> and T T Truong<sup>2</sup>

<sup>1</sup> Université Paris 7-Denis Diderot/LPTMC Tour 24-14.5ème étage, 4, Place Jussieu 75251 Paris Cedex 05, France

<sup>2</sup> Laboratoire de Physique Théorique et Modélisation Université de Cergy-Pontoise, F-95031 Cergy-Pontoise Cedex, France

E-mail: courbage@ccr.jussieu.fr

Received 22 July 2004, in final form 18 October 2004

Published 24 November 2004

Online at [stacks.iop.org/JPhysA/37/11759](http://stacks.iop.org/JPhysA/37/11759)

doi:10.1088/0305-4470/37/49/003

## Abstract

A duality formula, of the Hardy and Littlewood type for multidimensional Gaussian sums, is proved in order to estimate the asymptotic long-time behaviour of the distribution of the Birkhoff sums  $S_n$  of a sequence generated by a skew product dynamical system on the  $\mathbb{T}^2$  torus, with zero Lyapounov exponents. The sequence, taking the values  $\pm 1$ , is a pairwise independent (but not independent otherwise) ergodic sequence with infinite range dependence. The model corresponds to the motion of a particle on an infinite cylinder, hopping backward and forward along its axis, with a transversal acceleration parameter  $\alpha$ . We show that when the parameter  $\alpha/\pi$  is rational then all the moments of the normalized sums  $E((S_n/\sqrt{n})^k)$ , but the second, are unbounded with respect to  $n$ , while for irrational  $\alpha/\pi$ , with bounded continuous fraction representation, all these moments are finite and bounded with respect to  $n$ .

PACS numbers: 05.40.-a, 05.45.-a

## 1. Introduction

Diffusion processes are the most important transport processes to be modelled in the frame of dynamical systems. For a particle having deterministic ‘chaotic’ motion, diffusion appears in the long-time limit of the displacement governed by the dynamics under the invariant probability measure. The extension of the central limit theorem to deterministic dynamical systems is well established for many strongly chaotic systems. In these systems diffusion is related to strong mixing properties of the motion of the particles. However, this condition is sufficient, but not necessary.

A stationary bounded sequence of identically distributed random variables  $X_0, X_1, \dots$ , with zero mean value is ergodic if

$$\frac{1}{n} \sum_{i=0}^{n-1} X_i \xrightarrow[n \rightarrow \infty]{} E(X_0) = 0. \quad (1.1)$$

It fulfils the central limit theorem if there exists a positive sequence  $f_n$  such that  $\frac{1}{f_n} \sum_{i=0}^{n-1} X_i$  converges in law to a normal centred distribution with variance 1. The validity of the central limit theorem goes beyond the case of independent (Bernoulli) sequences  $X_i$ . Recently this problem has been cast in a more general setting of ergodic dynamical systems.

Let  $T$  be a measurable transformation on a space  $\mathcal{X}$ , equipped with a probability measure  $\mu$  on a  $\sigma$ -algebra  $\mathcal{F}$  of measurable sets. The measure  $\mu$  is supposed to be  $T$ -invariant: for all  $A$  of  $\mathcal{F}$  one has  $\mu(T^{-1}A) = \mu(A)$ . The system is also assumed to be ergodic:

$$\forall f \in \mathcal{L}^1(\mathcal{X}); \quad \frac{1}{n} \sum_{i=0}^{n-1} f(T^i X) \xrightarrow[n \rightarrow \infty]{} \int_{\mathcal{X}} f(x) d\mu(x) = E(f) \quad (1.2)$$

almost everywhere. The sum

$$S_n f(X) = \sum_{i=0}^{n-1} f(T^i X)$$

is called a *Birkhoff sum*. A real square-integrable function  $f$ , with zero mean value, is said to satisfy the central limit theorem if

$$\lim_{n \rightarrow \infty} \mu \left( X : \frac{1}{\sqrt{(S_n^2)}} \sum_{i=0}^{n-1} f(T^i X) \in I \right) = \frac{1}{\sqrt{2\pi}} \int_I \exp\left(-\frac{x^2}{2}\right) dx \quad (1.3)$$

In other words, the sum  $\frac{1}{\sqrt{(S_n^2)}} \sum_{i=0}^{n-1} f \circ T^i$  converges in law to a Gaussian variable of zero mean value and variance 1. A generalization of the central limit theorem has been established for a class of regular functions for  $K$ -systems [Go], and for hyperbolic dynamical systems the auto-correlations of which decrease exponentially (see a review in [Li]). One of the most remarkable results of the theory of chaotic dynamical systems was the convergence of normalized Birkhoff sums of observables to a diffusion process. In the Lorentz gas, Bunimovich, Chernov, Sinai [BCS], [BS] and Young [CYo] have proved such a behaviour. They found a class of functions  $f$  for which the rescaled ergodic sums converge to the Wiener process, i.e.

$$\sqrt{\tau} \sum_{n=0}^{[t/\tau]} f \circ T^n \xrightarrow[\tau \rightarrow 0]{} Y_t \quad (1.4)$$

where  $t \in [0, 1]$  and  $Y_t$  is a Brownian motion (other information can be found in [Bi]). In non-integrable area preserving maps numerical results on diffusion have been obtained by many authors (see, for instance, references in [Za], [MMP]). However to our knowledge, in the Hamiltonian dynamical systems no rigorous proofs have been given. In [Za], there are several numerical evidences of the existence of anomalous transport in Hamiltonian non integrable systems on account of the existence of islands into islands in the phase space of the systems due to the abundance of some stable periodic behaviours generating Levy flights. One of the main motivations of our work is to investigate diffusive behaviour in simple ergodic area-preserving mappings with weakly chaotic properties (i.e. randomness with zero Lyapounov exponents [CH3]). In such systems, the problem is to find a class of functions  $f$  for which

the rescaled ergodic sums converge to the Wiener process. A first step towards such a result is to estimate the asymptotic behaviour of the Birkhoff sums. In this work we consider this problem for completely non-hyperbolic systems having nevertheless a family of observables for which the two-times auto-correlations are zero. Swante Janson [J] has studied examples of nonergodic stationary sequences with zero mean value and variance  $\sigma$ , such that the sequence  $S_n = \sum_{i=0}^{n-1} X_i$  converges in law, without normalization, which implies a breaking of the central limit theorem.

We shall consider an ergodic sequence of pairwise independent variables, each of them taking values in  $\{-1, 1\}$  and having variance  $\sigma^2 = 1$ . They are derived from a dynamical system on the torus  $\mathbb{T}^2$ , called the Anzaï skew product and defined by

$$T(x, y) \equiv (x + \alpha, y + x) \pmod{2\pi}. \tag{1.5}$$

When  $\alpha/\pi$  is irrational, Furstenberg (see references in [CFS]) has shown that this transformation has a unique invariant measure, which is thus ergodic and is in fact the normalized Lebesgue measure  $\frac{dx dy}{(2\pi)^2}$ . In a previous work Courbage and Hamdan [CH1, CH2] have determined partitions  $\{A, A^C\}$ , which are pairwise independent:  $\mu(A \cap T^{-n}A) = \mu(A)^2$ . It is equivalent to say that for the function  $f = \mathbb{I}_A - \mu(A)$  the family of functions  $f \circ T^n$  are pairwise independent random variables. However, this does not imply at all that these variables are jointly independent. Actually, the process has infinite memory because its metric entropy is zero; moreover, there is only linear divergence of trajectories. It is to be noted that on account of the pairwise independence, it can be immediately seen that the ‘diffusion coefficient’  $\sigma$  defined by

$$\frac{1}{n} \int_I \left( \sum_{i=0}^{n-1} f(T^i x) \right)^2 dx \xrightarrow[n \rightarrow \infty]{} \sigma^2$$

is finite. However, the existence of the diffusion coefficient is not sufficient to imply asymptotic normal distribution.

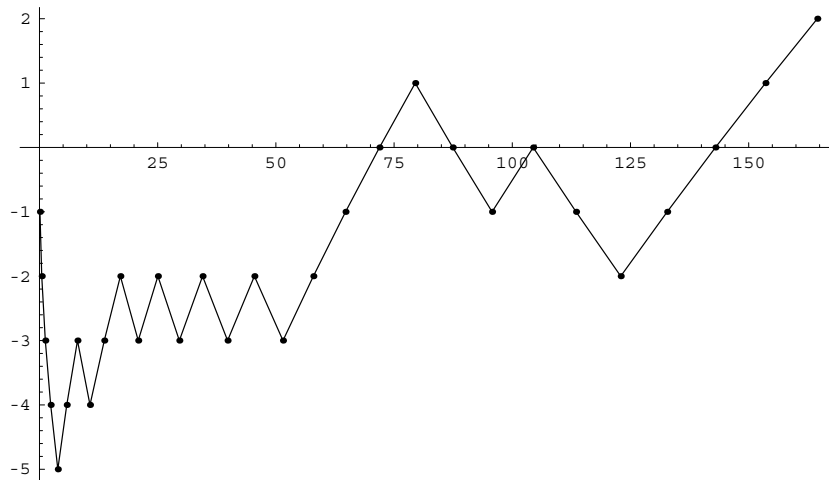
Thus it would be interesting to see whether the central limit theorem holds here or not. We shall see that the result depends essentially on the continued fraction representation of  $\alpha$ . In this work,  $A$  will be limited to the subset  $\mathbb{T} \times [0, \pi]$ ,  $X_i = f \circ T^i$  and the moments  $E((S_n^q))$  will be asymptotically estimated. It is however difficult to reconstruct the characteristic function  $\Phi_n(t)$  of  $\frac{S_n}{\sqrt{n}}$ . For odd  $q$ ,  $E(S_n^q) = 0$ . For even  $q$  several situations can occur. We shall show that

- (a) if  $\alpha/\pi$  is an irrational number with a bounded continued fraction representation, then  $E((S_n^{2q}) = \mathcal{O}(n^q)$ . Consequently  $|E((\frac{S_n}{\sqrt{n}})^{2q})| \leq A_q$ , where  $A_q$  is a constant. The convergence of the characteristic function depends on the  $A_q$ .
- (b) If  $\alpha/\pi$  is an irrational but not of the previous type, then there is no  $f_n$  such that  $E((\frac{S_n}{f_n})^{2k}) = \mathcal{O}(1)$ , different from zero for all  $k \geq 0$ .

It is to be noted that our mapping gives interesting examples where the existence of the diffusion coefficient  $\sigma^2 = \lim_{n \rightarrow \infty} E(S_n^2)/n = 1$  is not sufficient to imply the convergence to the normal distribution.

So it may be suggestive to start out with a concrete picture of particle dynamics. The dynamical system (1.5) corresponds to the motion of a rotator kicked at a regular time interval by a force modulated so that the angular velocity remains bounded. Let  $\theta(t)$  and  $\dot{\theta}(t)$  be the angle and the angular velocity of the rotator, where  $\dot{\theta}(t) \in [0, 2\pi[$ . Then, at time  $t + 1$ , the state of the rotator is given by

$$\begin{aligned} \theta(t + 1) &\equiv \theta(t) + \dot{\theta}(t) \pmod{2\pi} \\ \dot{\theta}(t + 1) &\equiv \dot{\theta}(t) + \alpha \pmod{2\pi} \end{aligned} \tag{1.6}$$



**Figure 1.** Trajectory of the particle in the plane  $(q_1, q_2)$ .

and we have at time  $t$ :

$$\theta(t) \equiv \theta(0) + t\dot{\theta}(0) + \frac{t(t-1)\alpha}{2} \pmod{2\pi}. \quad (1.7)$$

An idea of the randomness of this motion can be seen by using a partition  $\{A, A^C\}$  of the torus into two regions:  $A = \{\theta \in [0, \pi[ ]$  and  $A^C = \{\theta \in [\pi, 2\pi[ ]$ . It can be seen that

$$\mu(A_i \cap T^{-n}A_j) = \mu(A_i)\mu(A_j)$$

for any  $n \neq 0$ , where  $A_i$  is either  $A$  or  $A^c$  [CH1]. This partition is far from being a Bernoulli one since the entropy of the system is zero. We call such partitions pairwise independent. In [CH3], it is shown that such sequences are unpredictable in the sense of the Wiener least squares criterion. It is natural to study the distributions of sums of such sequences, which represent a particle displacement induced by the dynamical system of equation (1.5) as follows.

A particle moves on an infinite plane among periodically distributed obstacles with spatial period equal to 1 along both  $(q_1, q_2)$ -directions. In the  $q_1$ -direction, the motion of the particle is uniformly accelerated at each regular time interval by an amount  $\alpha$  and has uniform free motion along the  $q_2$ -direction. That is, define the velocity  $p_1(n) = q_1(n+1) - q_1(n)$ , then the equations of the projection of the motion in the  $q_1$ -direction are

$$q_1(n) = q_1(n-1) + p_1(n-1) \quad (1.8)$$

$$p_1(n) = p_1(n-1) + \alpha. \quad (1.9)$$

It is the result of the discrete time action of the mapping:  $T : (p_1, q_1) \rightarrow T(p_1, q_1)$  given by

$$T(p_1, q_1) = (p_1 + \alpha, q_1 + p_1). \quad (1.10)$$

Thus, we obtain

$$q_1(n) = q_1(0) + np_1(0) + n(n-1)\alpha/2. \quad (1.11)$$

The particle is moreover submitted at the beginning of each time interval to a field changing the direction of the motion up and down along the  $q_2$ -direction in terms of its position along the  $q_1$ -direction in the following way (see figure 1): the velocity direction of the

particle  $p_2$  at time  $t = n$  is given by  $\chi(q_1(n))$  where  $\chi(x)$  is a periodic discontinuous function defined by

$$\chi(x) = \begin{cases} -1 & \text{if } x \in ]0, 1/2] \\ 1 & \text{if } x \in ]1/2, 1]. \end{cases} \tag{1.12}$$

That is, the direction of the velocity at time  $t = n$  is

$$\varepsilon_n(p_1(0), q_1(0)) = \chi(q_1(n)) = \chi(q_1(0) + np_1(0) + n(n - 1)\alpha/2) \tag{1.13}$$

and the value of the variable  $q_2$  at time  $t = n + 1$  is

$$q_2(n + 1) = q_2(n) + \chi(q_1(n)) = \sum_1^n \varepsilon_i(p_1(0), q_1(0)) \tag{1.14}$$

Let us denote by  $S_k$  the position of the particle along the  $q_2$ -direction at  $t = k$ . The equation for such a motion is expressed by the following recursion relation:

$$S_{n+1} = S_n + \chi(q_1(n)) \quad S_0 = 0. \tag{1.15}$$

In terms of the variables  $(x, y) \in \mathbb{T}^2$  the distance travelled by the particle along the vertical direction after  $n$  steps is

$$S_n(x, y; \alpha) = \sum_{k=0}^{n-1} \chi(y(k)) = \sum_{k=0}^{n-1} \chi\left(y + kx + \frac{k(k-1)}{2}\alpha\right). \tag{1.16}$$

The problem is to study the limiting distribution of such a random walk.

In the following sections, the proof will be given in steps with an increasing order of complexity, which turns out to be related to number theoretical questions. The main objective is to obtain the asymptotic behaviour of the expectation values of  $S_n^q(\alpha)$  for  $n \rightarrow \infty$ . In the course of this study, we encounter the so-called multidimensional Gaussian sums which are a particular case of the Weyl sums [K]. They are essentially sums of the form used in the definition of theta functions, except that the summation is finite. The Gaussian sums fulfil an exact duality formula, known as Landsberg–Schaar formula [L], which is a variant of the famous Jacobi identity for theta functions. Except for the rational assumptions in the application of the Schaar formula, some works had been done to estimate the growth rate of the one-dimensional Gaussian sums by Hardy and Littlewood in [HL]. To this end they have first established a duality formula and then used the theory of continuous fraction decomposition (here we use tools and notation from [VdP] and [R]). We shall treat respectively in sections 3, 4 and 5 Gaussian sums in one, two and  $d$  dimensions, the central point in each part is concentrated on the proof of a duality formula for these sums.

The Gaussian sums have been more recently studied from a geometrical point of view. The results of [HL] on polynomials associated with modular functions have been applied in [DM] and [Z]. Interestingly one may derive this identity from a simple quantum mechanical system having the torus as phase space [AR]. But Gaussian sums are found also in other fields such as the problem of fractional wave packet revival [SLB].

## 2. Preliminary considerations and the case of $\alpha$ rational

Before going into the crux of the subject, let us introduce some technical steps. The important quantity to study here is the  $q$ th moment defined by

$$E(S_n^q(\alpha)) = \int_{[0, 2\pi[ \times [0, 2\pi[} (S_n^q(x, y; \alpha)) \frac{dx dy}{4\pi^2}. \tag{2.1}$$

An objective is to study the convergence of the characteristic function

$$\Phi_n(t) = \int_{[0, 2\pi] \times [0, 2\pi]} e^{it(S_n(x, y; \alpha)/\sqrt{n})} \frac{dx dy}{4\pi^2} = \sum_{q=0}^{\infty} \frac{(it)^q}{q!} E(S_n^q(\alpha)/\sqrt{n}). \tag{2.2}$$

Using the Fourier decomposition of the function  $\chi$ ,

$$\chi(y) = \frac{2i}{\pi} \sum_{p \in \tilde{\mathbb{Z}}} \frac{e^{iPy}}{P} \tag{2.3}$$

where  $\tilde{\mathbb{Z}}$  is the set of relative odd integers, we have an alternative form of the distance travelled by the particle

$$S_n(x, y; \alpha) = \frac{2i}{\pi} \sum_{P \in \tilde{\mathbb{Z}}} \sum_{k=0}^{n-1} \frac{\exp(iP(y + kx + \frac{k(k-1)}{2}\alpha))}{P}. \tag{2.4}$$

Hence with this form one may compute the  $q$ th moment. After integration on  $x$  and  $y$ , which yields two constraints

$$\sum_{i=1}^q P_i = 0 \quad \sum_{i=1}^q k_i P_i = 0, \tag{2.5}$$

the above expression becomes

$$E[S_n^q(\alpha)] = \left(\frac{2i}{\pi}\right)^q \sum_{P_1 \in \tilde{\mathbb{Z}}} \dots \sum_{P_q \in \tilde{\mathbb{Z}}} \sum_{k_1=0}^{n-1} \dots \sum_{k_q=0}^{n-1} \frac{1}{P_1 \dots P_q} \exp(i2\pi\beta(P_1 k_1^2 + \dots + P_q k_q^2)) \tag{2.6}$$

where  $\beta = \frac{\alpha}{4\pi}$ . We are now in a position to prove

**Lemma 2.1.**  $E[S_n^{2q+1}(\alpha)] = 0$

$$E[S_n^{2q}(\alpha)] = (-1)^q \frac{4^q}{\pi^{2q}} \sum_{P_1 \in \tilde{\mathbb{Z}}} \dots \sum_{P_{2q} \in \tilde{\mathbb{Z}}} \sum_{k_1=0}^{n-1} \dots \sum_{k_{2q}=0}^{n-1} \frac{\delta(\sum_{i=2}^{2q} P_i(k_i - k_1))}{P_1 \dots P_{2q}} \times \exp i\pi\beta \sum_{i=2}^{2q} \sum_{j=2}^{2q} \frac{P_i P_j}{\sum_{l=2}^{2q} P_l} (k_i - k_j)^2 \tag{2.7}$$

where  $\delta(\cdot)$  is the Kronecker function: equal to 1 when the argument is 0 and 0 otherwise.

**Proof.**  $(P_i)_{i=1, \dots, 2q}$  are odd. The condition  $\sum_{i=1}^q P_i = 0$  can only be satisfied if  $q$  is even. Hence  $E[S_n^{2q+1}(\alpha)] = 0$ . Now from the constraint  $\sum_{i=1}^{2q} P_i = 0$ , we get  $P_1 = -\sum_{i=2}^{2q} P_i$  and from  $\sum_{i=1}^{2q} k_i P_i = 0$  i.e.  $\sum_{i=2}^{2q} P_i(k_i - k_1) = 0$ . Thus, working out  $k_1$ :

$$k_1 = \frac{\sum_{i=2}^{2q} P_i k_i}{\sum_{i=2}^{2q} P_i} \tag{2.8}$$

with  $(k_i)_{i=2, \dots, n-1} \in \{0, \dots, n-1\}$ . And we obtain, by substitution

$$\begin{aligned} \sum_{i=1}^{2q} P_i k_i^2 &= \frac{1}{\sum_{i=2}^{2q} P_i} \left( \frac{1}{2} \sum_{i=2}^{2q} \sum_{j=2}^{2q} P_i P_j (k_i^2 + k_j^2) - \sum_{i=2}^{2q} \sum_{j=2}^{2q} P_i P_j k_i k_j \right) \\ &= \frac{1}{\sum_{i=2}^{2q} P_i} \left( \frac{1}{2} \sum_{i=2}^{2q} \sum_{j=2}^{2q} P_i P_j (k_i - k_j)^2 \right) \end{aligned} \tag{2.9}$$

which ends the proof. □

In particular  $E[S_n^2(\alpha)]$  can be exactly and directly computed:

$$E[S_n^2(\alpha)] = \frac{4}{\pi^2} \sum_{P_1 \in \mathbb{Z}} \sum_{P_2 \in \mathbb{Z}} \sum_{k_1=0}^{n-1} \sum_{k_2=0}^{n-1} \frac{\exp(-i2\pi\beta(P_1 k_1^2 + P_2 k_2^2))}{P_1 P_2}. \tag{2.10}$$

But we get from the constraints,  $P_1 = -P_2 = P$  and  $k_1 = k_2 = k$ . One has thus

$$E[S_n^2(\alpha)] = \frac{4}{\pi^2} \sum_{P \in \mathbb{Z}} \frac{1}{P^2} \sum_{k=0}^{n-1} 1 \tag{2.11}$$

From the property of the Riemann zeta-function [MOS], it may be shown that  $\sum_{P \in \mathbb{Z}} 1/P^2 = \pi^2/4$ . Hence

$$E[S_n^2(\alpha)] = n. \tag{2.12}$$

The fact that odd moments are zero implies that the characteristic function is also even and consequently the Fourier inverse transformation of the characteristic function, that is the probability distribution function of the random variable  $S_n(x, y; \alpha)$  is also even.

Finally we observe that the series

$$\sum_{(P_2, \dots, P_{2q}) \in \mathbb{Z}^{2q-1}} \frac{1}{P_2 \cdots P_{2q} (P_2 + \cdots + P_{2q})} \tag{2.13}$$

is absolutely convergent since for all  $(P_2, \dots, P_{2q}) \in \mathbb{Z}^{2q-1}, |P_2 + \cdots + P_{2q}| \geq 1$ . Let us prove it for  $q = 2$ . The generalization to an arbitrary  $q$  is straightforward. The series may be split into

$$\begin{aligned} \sum_{(P_2, P_3, P_4) \in \mathbb{Z}^3} \frac{1}{|P_2 P_3 P_4 (P_2 + P_3 + P_4)|} &= 2 \sum_{(P_2, P_3, P_4) \in \mathbb{N}^3} \frac{1}{|P_2 P_3 P_4 (P_2 + P_3 + P_4)|} \\ &+ 6 \sum_{(P_2, P_3, P_4) \in \mathbb{N}^3} \frac{1}{|P_2 P_3 P_4 (P_2 + P_3 - P_4)|}. \end{aligned} \tag{2.14}$$

Let  $(x, y, z) \in [0, \infty]^3$  and  $0 < \delta P < 1$  such that

$$\begin{aligned} 1 \leq P_2 \leq x < P_2 + \delta P &\Leftrightarrow x - \delta P < P_2 \leq x \\ 1 \leq P_3 \leq y < P_3 + \delta P &\Leftrightarrow y - \delta P < P_3 \leq y \\ 1 \leq P_4 \leq z < P_4 + \delta P &\Leftrightarrow z - \delta P < P_4 \leq z. \end{aligned} \tag{2.15}$$

The first sum on the rhs of the equation (1.16) is bounded

$$\begin{aligned} \sum_{(P_2, P_3, P_4) \in \mathbb{N}^3} \frac{1}{|P_2 P_3 P_4 (P_2 + P_3 + P_4)|} &\leq \frac{1}{6^{1/3}} \sum_{(P_2, P_3, P_4) \in \mathbb{N}^3} \frac{1}{|P_2 P_3 P_4 (P_2 P_3 P_4)^{1/3}|} \\ &= \frac{1}{6^{1/3}} \left( \sum_{(P) \in \mathbb{N}} \frac{1}{P^{1+1/3}} \right)^3 < \infty. \end{aligned} \tag{2.16}$$



The second sum can be transformed using  $0 < (x - \delta P)(y - \delta P)(z - \delta P) \leq P_2 P_3 P_4 \leq xyz$  and  $x + y - z - 2\delta P \leq P_2 + P_3 - P_4 \leq x + y - z + \delta P$ . Hence

$$\frac{1}{|P_2 P_3 P_4 (P_2 + P_3 - P_4)|} \leq \frac{1}{(x - \delta P)(y - \delta P)(z - \delta P)} \times \max\left(\frac{1}{|x + y - z - 2\delta P|}, \frac{1}{|x + y - z + \delta P|}\right) \stackrel{\text{def}}{=} F_{\delta P}(x, y, z). \tag{2.17}$$

But  $\forall \delta P < 1, \exists \epsilon > 0$ , such that any hypercube

$$[(P_2, P_3, P_4), \dots, (P_2 + \delta P, P_3 + \delta P, P_4 + \delta P)]$$

does not lie in the intersection  $\Omega_\epsilon \stackrel{\text{def}}{=} [0, \infty[^3 \cap \{|x + y - z| > \epsilon\}$ . Then we have

$$\sum_{(P_2, P_3, P_4) \in \mathbb{N}^3} \frac{1}{|P_2 P_3 P_4 (P_2 + P_3 - P_4)|} \leq \frac{1}{(\delta P)^3} \sum_{(P_2, P_3, P_4) \in \mathbb{N}^3} \int_{P_2}^{P_2 + \delta P} \int_{P_3}^{P_3 + \delta P} \int_{P_4}^{P_4 + \delta P} F_{\delta P}(x, y, z) \leq \frac{1}{(\delta P)^3} \int_{[0, \infty[^3 \setminus \Omega_\epsilon} F_{\delta P}(x, y, z). \tag{2.18}$$

One can easily estimate the left-hand side by realizing that the integration of the integrals

$$\int_{[0, \infty[^3 \setminus \Omega_\epsilon} \frac{dx dy dz}{|x + y - z + \delta P|} \quad \text{and} \quad \int_{[0, \infty[^3 \setminus \Omega_\epsilon} \frac{dx dy dz}{|x + y - z - 2\delta P|} \tag{2.19}$$

yields an absolutely convergent result.

2.1.  $\alpha = 0$

We consider the simple case  $\alpha = 0$  and show that the case with rational  $\alpha$  can be brought back to the case  $\alpha = 0$ . First consider again

$$S_n(x, y; 0) = \frac{2}{i\pi} \sum_{P \in \mathbb{Z}} \frac{1}{P} \sum_{k=0}^{n-1} e^{iP(kx+y)} \tag{2.20}$$

The above geometric sum can be re-expressed in terms of  $u = e^{ix}$  and  $v = e^{iy}$  as

$$S_n(x, y; 0) = \frac{2}{i\pi} \sum_{P \in \mathbb{Z}} \frac{1}{P} z^{-\frac{n-1}{2}P} \left(\frac{1 - u^{nP}}{1 - u^P}\right) \sum_{\epsilon = \pm 1} \epsilon u^{\epsilon P \frac{n-1}{2}} v^{\epsilon P} \tag{2.21}$$

Now, the  $2k$ th moment

$$E[S_n^{2k}(0)] = \int_{[0, 2\pi[ \times [0, 2\pi[} \left[ \frac{2}{i\pi} \sum_{P \in \mathbb{Z}} \frac{1}{P} \left( \sum_{k=0}^{n-1} e^{-iP(kx+y)} \right) \right]^{2k} \frac{dx dy}{4\pi^2} \tag{2.22}$$

may be computed as complex integrals evaluated on the the product of the circles of unit radius and centred at 0, i.e.  $\mathcal{C}(0, 1) \times \mathcal{C}(0, 1)$ . In the domain bounded by these circles, the integrand is biholomorph and the product of the contours is homotopical to  $\mathcal{C}(0, 1/2) \times \mathcal{C}(0, 1/2)$ . Thus  $(1 - u^P)^{-1}$  can be expanded in power series around 0, with  $u = e^{ix}$  and  $v = e^{iy}$ :

$$S_n(u, v; 0) = \frac{2i}{\pi} \sum_{P \in \mathbb{Z}} u^{\frac{P}{2}} \left( \sum_{\eta = \pm 1} \eta u^{\eta \frac{P}{2}} \right) \left( \sum_{m=0}^{\infty} u^{mP} \right) \left( \sum_{\epsilon = \pm 1} \frac{\epsilon}{P} u^{\epsilon \frac{n-1}{2} P} v^{\epsilon P} \right) \tag{2.23}$$

or

$$S_n(u, v; 0) = \frac{2i}{\pi} \sum_{\substack{\eta=\pm 1 \\ \epsilon=\pm 1}} \sum_{\substack{m \in \mathbb{N} \\ P \in \tilde{\mathbb{N}}} } \frac{\eta \epsilon}{P} u^{P(\frac{\epsilon+\eta}{2}n+m+\frac{1-\epsilon}{2})} v^{\epsilon P}. \tag{2.24}$$

We can now compute  $E[S_n^2(0)]$  with the expression

$$\begin{aligned} E[S_n^2(0)] &= -\frac{4}{\pi^2} \sum_{\substack{\eta_1, \eta_2 = \pm 1 \\ \epsilon_1, \epsilon_2 = \pm 1}} \sum_{\substack{m_1, m_2 \in \mathbb{N} \\ P_1, P_2 \in \tilde{\mathbb{N}}}} \frac{\eta_1 \epsilon_1}{P_1} \frac{\eta_2 \epsilon_2}{P_2} \oint \frac{dv}{2i\pi} v^{\epsilon_1 P_1 + \epsilon_2 P_2} \\ &\quad \times \oint \frac{du}{2i\pi} u^{P_1(\frac{\epsilon_1+\eta_1}{2}n+m_1+\frac{1-\epsilon_1}{2})+P_2(\frac{\epsilon_2+\eta_2}{2}n+m_2+\frac{1-\epsilon_2}{2})}. \end{aligned} \tag{2.25}$$

The first integral over  $v$  yields  $\epsilon_1 P_1 + \epsilon_2 P_2 = 0$ , one must have  $\epsilon_1 = -\epsilon_2$  and  $P_1 = P_2 = P$ . The exponent of  $z$  in the second integral becomes

$$\begin{aligned} &P_1 \left( \frac{\epsilon_1 + \eta_1}{2}n + m_1 + \frac{1 - \epsilon_1}{2} \right) + P_2 \left( \frac{\epsilon_2 + \eta_2}{2}n + m_2 + \frac{1 - \epsilon_2}{2} \right) \\ &= P \left( \frac{\eta_1 + \eta_2}{2}n + m_1 + m_2 + 1 \right). \end{aligned} \tag{2.26}$$

The contribution to the complex integration comes only from the vanishing of this quantity. This leads to the following:

$$\eta_1 = \eta_2 = -1 \tag{2.27}$$

$$m_1 + m_2 = n - 1 \tag{2.28}$$

and  $\text{card} \{(m_1, m_2) \in \mathbb{N}^2 \text{ such that } m_1 + m_2 = n - 1\} = n$ , thus

$$E[S_n^2(0)] = \sum_{\epsilon=\pm 1} \sum_{P \in \tilde{\mathbb{N}}} \frac{4n}{P^2 \pi^2} = \frac{4n}{\pi^2} \sum_{P \in \tilde{\mathbb{Z}}} \frac{1}{P^2} = n. \tag{2.29}$$

This computational method may be extended to higher moments  $E[S_n^{2k}(0)]$ . We see that using equation (2.25) to power  $2k$  yields

$$\begin{aligned} E[S_n(0)^{2k}] &= (-1)^k \left( \frac{4}{\pi^2} \right)^k \sum_{\eta_1, \dots, \eta_{2k} = \pm 1} \sum_{\epsilon_1, \dots, \epsilon_{2k} = \pm 1} \sum_{m_1, \dots, m_{2k} \in \mathbb{N}} \sum_{P_1, \dots, P_{2k} \in \tilde{\mathbb{N}}} \\ &\prod_{i=1}^{2k} \frac{\eta_i \epsilon_i}{P_i} \oint \frac{dv}{2i\pi} v^{(\sum_{i=1}^{2k} \epsilon_i P_i)} \oint \frac{du}{2i\pi} u^{(\sum_{i=1}^{2k} P_i (m_i + \frac{1-\epsilon_i}{2} + \frac{\eta_i + \epsilon_i}{2}n))}. \end{aligned} \tag{2.30}$$

We can perform the  $v$ -integration and get

$$\sum_{i=1}^{2k} \epsilon_i P_i = 0 \tag{2.31}$$

and the  $u$ -integration yields the following proposition:

**Proposition 1.**

$$\begin{aligned} E[S_n(0)^{2k}] &= (-1)^k \left( \frac{4}{\pi^2} \right)^k \sum_{\eta_1, \dots, \eta_{2k} = \pm 1} \sum_{\epsilon_1, \dots, \epsilon_{2k} = \pm 1} \sum_{m_1, \dots, m_{2k} \in \mathbb{N}} \sum_{P_1, \dots, P_{2k} \in \tilde{\mathbb{N}}} \\ &\prod_{i=1}^{2k} \frac{\eta_i \epsilon_i}{P_i} A_n[\eta_i, \epsilon_i, P_i](2k) \end{aligned} \tag{2.32}$$

where the coefficients  $A_n$  are defined as

$$A_n[\eta_i, \epsilon_i, P_i](2k) = \text{card} \left\{ (m_1, \dots, m_{2k}) \in \mathbb{N}^{2k} \left| \sum_{i=1}^{2k} m_i P_i = -\frac{1}{2} \sum_{i=1}^{2k} P_i (1 + n \eta_i) \right. \right\} \quad (2.33)$$

with the constraint  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$ .

This result is obtained by the *residue theorem*. It remains to determine the behaviour of  $A_n$  as  $n \rightarrow \infty$ . We observe that not all the combinations of  $\epsilon_i$  are to be taken into account since  $(P_1, \dots, P_n) \in \tilde{\mathbb{N}}^{2k}$ . Thus the choice  $(\epsilon_1, \dots, \epsilon_{2k}) = (1, \dots, 1)$  is excluded by the constraint. According to the usual notation, if the  $(P_i)_{i=1, \dots, r}$  do not have common divisors, we note  $(P_1, \dots, P_r) = 1$ , then

$$\text{card} \left\{ (m_1, \dots, m_r) \in \mathbb{N}^r \left| \sum_{i=1}^r m_i P_i = n \right. \right\} \underset{n \rightarrow \infty}{\sim} \frac{1}{(r-1)!} \frac{n^{r-1}}{P_1 \cdots P_r} \quad (2.34)$$

Before going on we prove the following lemma:

**Lemma 2.** Let  $\sum_{i=1}^r \epsilon_i P_i = 0$  and  $(P_1, \dots, P_r) = 1$  is equivalent to  $(P_1, \dots, \hat{P}_j, \dots, P_r) = 1$  where  $P_j$  is removed.

**Proof.** It is clear that if  $(P_1, \dots, \hat{P}_j, \dots, P_r) = 1$  then  $(P_1, P_r) = 1$ . Actually if  $(P_1, \dots, \hat{P}_j, \dots, P_r) = 1$  then  $\exists s \in \tilde{\mathbb{N}} \setminus \{1\}$  such that  $\forall i \neq j, P_i = s p_i$  and thus  $(P_1, \dots, P_q) = 1$ .

Conversely if  $(P_1, \dots, P_q) = 1$ , but  $(P_1, \dots, \hat{P}_j, \dots, P_r) = s$  with  $s \in \tilde{\mathbb{N}} \setminus \{1\}$ , i.e.  $\forall i \neq j, P_i = s p_i$ , then  $P_j = \sum_{i \neq j} \epsilon_i \epsilon_j P_i = s (\sum_{i \neq j} \epsilon_i \epsilon_j p_i)$  which contradicts the fact that  $(P_1, \dots, P_q) = 1$ .  $\square$

Coming back to our estimate of  $A_n$ , put  $\tilde{\mathbb{N}}^q = \tilde{\mathbb{N}}^q \cap \{(P_1, \dots, P_q) = 1\}$ . If  $(P_1, \dots, P_{2k}) \in \tilde{\mathbb{N}}^{2k}$  such that  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$  then

$$A_n[\eta_i, \epsilon_i, P_i](2k) \underset{n \rightarrow \infty}{\sim} \frac{(-1)^{2k-1}}{(2k-1)!} \frac{1}{\prod_{i=1}^{2k} P_i} \left( \frac{n \cdot \sum_{i=1}^{2k} P_i \eta_i}{2} \right)^{2k-1} \quad \text{if } \sum_{i=1}^{2k} \eta_i P_i < 0$$

$$= \begin{cases} 1 & \text{if } \sum_{i=1}^{2k} \eta_i P_i = 0 \\ 0 & \text{otherwise} \end{cases} \quad (2.35)$$

with  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$ . Now from equation (2.35) the asymptotic behaviour for  $n \rightarrow \infty$  of the  $2k$ th moment is

$$E[S_n^{2k}(0)] \underset{n \rightarrow \infty}{\sim} 2 \frac{(-1)^k}{(2k-1)! \pi^{2k}} \sum_{P \in \tilde{\mathbb{N}}} \frac{1}{P^{2k}} \sum_{\substack{\eta_i = \pm 1 \\ \epsilon_i = \pm 1}} \sum_{P_i \in \tilde{\mathbb{N}}^{2k}} \prod_{i=1}^{2k} \frac{\epsilon_i \eta_i}{P_i^2} \left( \sum_{i=1}^{2k} \eta_i P_i \right)^{2k-1} n^{2k-1} \quad (2.36)$$

with  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$  and  $\sum_{i=1}^{2k} \eta_i P_i > 0$ . We have already seen that the series

$$\sum_{\substack{\eta_1, \dots, \eta_{2k} = \pm 1 \\ \epsilon_1, \dots, \epsilon_{2k} = \pm 1}} \sum_{\substack{P_1, \dots, P_{2k} \in \tilde{\mathbb{N}}^{2k} \\ \sum_{i=1}^{2k} \epsilon_i P_i = 0}} \prod_{i=1}^{2k} \frac{\epsilon_i \eta_i}{P_i^2} \left( \sum_{i=0}^{2k} \eta_i P_i \right)^{2k-1} \quad (2.37)$$

is absolutely convergent. Since  $\sum_{P \in \mathbb{N}} \frac{1}{P^q} \xrightarrow{q \rightarrow \infty} 1$ , we have thus the theorem:

**Theorem 2.2.**  $\forall k \in \mathbb{N}, E[S_n^{2k}(0)] \underset{n \rightarrow \infty}{\sim} C_k n^{2k-1}$  with

$$C_1 = 1$$

$$C_k = 2 \frac{(-1)^k}{(2k-1)! \pi^{2k}} \sum_{\substack{\eta_i, \epsilon_i = \pm 1 \\ P_i \in \mathbb{N}^{2k}}} \prod_{i=1}^{2k} \frac{\epsilon_i \eta_i}{P_i^2} \left( \sum_{i=1}^{2k} \eta_i P_i \right)^{2k-1} \tag{2.38}$$

with  $\sum_{i=0}^{2k} \epsilon_i P_i = 0$  and  $\sum_{i=0}^{2k} \eta_i P_i > 0$ .

2.2. Extension to rational  $\beta$  ( $\alpha \in \mathbb{Q}$ )

To this end, we regroup terms of  $S_n(x, y; \alpha)$ .  $\forall \alpha \in \mathbb{Q}, \exists k_0 \in \mathbb{N}$  such that  $\forall k < k_0$ :

$$\begin{aligned} k(k-1)\beta &\not\equiv 0 \pmod{1} \\ k_0(k_0-1)\beta &\equiv 0 \pmod{1}. \end{aligned} \tag{2.39}$$

We shall set  $n = qk_0$  and write  $S_n(x, y; \alpha)$  grouping the terms such that

$$\delta_s(x, y) = \sum_{n=0}^{q-1} \chi(y + [s + nk_0]x + 2s(s-1)\pi\beta) \tag{2.40}$$

and

$$S_n(x, y; \alpha) = \sum_{s=0}^{k_0-1} \delta_s(x, y) \tag{2.41}$$

with  $0 < s \leq k_0$ . The sum can be partially performed after inserting the definition of  $\chi$ :

$$\begin{aligned} \delta_s(x, y) &= \frac{2}{i\pi} \sum_{P \in \mathbb{Z}} \frac{1}{P} e^{-ik_0 \frac{P(P-1)}{2} x} \left( \frac{e^{iPqk_0x} - 1}{e^{iPk_0x} - 1} \right) \\ &\quad \times \left( \exp \left( iP \left( y + x \left[ s + \frac{k_0(q-1)}{2} \right] + 2s(s-1)\beta \right) \right) \right. \\ &\quad \left. - \exp \left( -iP \left( y + x \left[ s + \frac{k_0(q-1)}{2} \right] + 2s(s-1)\beta \right) \right) \right). \end{aligned} \tag{2.42}$$

Call now  $s' = 2s(s-1), u = e^{ix}, v = e^{iy}$  and  $t = e^{2i\beta}$ . Then

$$\delta_s(x, y; \alpha) = \delta_s(u, v; t) = \frac{2}{i\pi} \sum_{P \in \mathbb{Z}} \frac{1}{P} u^{-k_0 \frac{P(P-1)}{2}} \left( \frac{1 - z^{Pqk_0}}{1 - z^{Pk_0}} \right) \sum_{\epsilon = \pm 1} \epsilon z^{\epsilon P(s + \frac{q-1}{2}k_0)} v^{\epsilon P} t^{\epsilon Ps'}. \tag{2.43}$$

We now expand  $\frac{1}{1-u^{Pk_0}}$  as in the previous section, and obtain consequently

$$\delta_s(u, v; t) = \frac{2}{i\pi} \sum_{\epsilon, \eta = \pm 1} \sum_{P \in \mathbb{Z}, m \in \mathbb{N}} \frac{\epsilon \eta}{P} v^{\epsilon P} t^{\epsilon Ps'} u^{P(\epsilon s + k_0 \frac{1-\epsilon}{2} + mk_0 + qk_0 \frac{\eta + \epsilon}{2})} \tag{2.44}$$

The expectation value

$$\begin{aligned} E(\delta_{s_1}(t) \dots \delta_{s_{2k}}(t)) &= (-1)^q \left( \frac{4}{\pi} \right)^q \sum_{\substack{\eta_i, \epsilon_i = \pm 1 \\ P_i \in \mathbb{N}, m_i \in \mathbb{N}}} \prod_{i=1}^{2k} \frac{\eta_i \epsilon_i}{P_i} t^{\left( \sum_{i=1}^{2k} \epsilon_i P_i s'_i \right)} \\ &\quad \times \oint \frac{dv}{2i\pi v} v^{\left( \sum_{i=1}^{2k} \epsilon_i P_i \right)} \oint \frac{du}{2i\pi u} u^{\left( \sum_{i=1}^{2k} P_i \left( \epsilon_i s_i + k_0 \frac{1-\epsilon_i}{2} + m_i k_0 + qk_0 \frac{\eta_i + \epsilon_i}{2} \right) \right)} \end{aligned} \tag{2.45}$$

is not zero if  $\sum_{i=1}^{2k} P_i (\epsilon_i s_i + k_0 \frac{1-\epsilon_i}{2} + m_i k_0 + q k_0 \frac{\eta_i + \epsilon_i}{2}) = 0$  and  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$ . As before, one can reduce to the case  $(P_1, \dots, P_{2k}) = 1$ . The first assertion is true if  $\sum_{i=0}^{2k_0} \epsilon_i P_i s_i$  is divisible by  $k_0$ , with  $0 \leq s_i < k_0$ . The hypothesis on  $P_i$  suggests that there exists a  $P_{i_0}$  not divisible by  $k_0$ .

**Lemma 3.** *Let  $s_1, \dots, s_{2k}$  such that  $\sum_{i=0}^{2k_0} \epsilon_i P_i s_i$  is divisible by  $k_0$  then  $\forall s \in \{1, \dots, k_0\}$  and  $s \neq s_{i_0}$ ,  $k_0$  is not a divisor of  $\sum_{i=0}^{2k_0} \epsilon_i P_i s_i$ .*

**Proof.** Suppose that this assertion is not true:  $\exists n$  such that  $\sum_{i=0}^{2k_0} \epsilon_i P_i s_i = nk_0$  and  $\exists n'$  such that  $\sum_{i=0}^{2k_0} \epsilon_i P_i s_i = n'k_0$ . Hence  $\epsilon_{i_0} P_{i_0} (s_{i_0} - s) = (n - n')k_0$  but  $k_0$  is not a divisor of  $P_{i_0}$  and  $|s_{i_0} - s| < k_0$ , this is not possible.  $\square$

Consequently  $\text{card}\{(s_1, \dots, s_{2k}) \in \{1, \dots, k_0\}^{2k} \text{ such that } k_0 \text{ is not a divisor of } \sum_{i=1}^{2k} \epsilon_i P_i s_i \text{ and the value } (P_i)_{i=1, \dots, 2k} \text{ are mutually prime numbers of values less than } k_0^{2k-1}\}$ .

**Remark 4.** We have

$$\text{card} \left[ (m_1, \dots, m_{2k}) \in \mathbb{N}^{2k} \left| \sum_{i=1}^{2k} P_i \left( \epsilon_i s_i + k_0 \frac{1-\epsilon_i}{2} + m_i k_0 + q k_0 \frac{\eta_i + \epsilon_i}{2} \right) = 0 \right. \right] \\ \underset{q \rightarrow \infty}{\simeq} \text{card} \left[ (m_1, \dots, m_{2k}) \in \mathbb{N}^{2k} \left| \sum_{i=1}^{2k} m_i P_i = -\frac{q}{2} \sum_{i=1}^{2k} \eta_i P_i \right. \right] \tag{2.46}$$

with  $(P_1, \dots, P_{2k}) = 1$ . Since  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$  and  $t^P \sum_{i=1}^{2k} \epsilon_i P_i s_i' = t^P \sum_{i=1}^{2k} \epsilon_i P_i s_i'^2 / 2$ . Following the previous procedure we obtain for  $n = qk_0$ .

$$E[S_{qk_0}^{2k}(\alpha)] \underset{q \rightarrow \infty}{\simeq} \frac{2(-1)^k}{\pi^{2k}(2k-1)!} (qk_0)^{2k-1} \sum_{\substack{\eta_i, \epsilon_i = \pm 1 \\ P_i \in \mathbb{N}}} \prod_{i=1}^{2k} \frac{\eta_i \epsilon_i}{P_i^2} \left( \sum_{i=1}^{2k} \eta_i P_i \right)^{2k-1} \frac{1}{k_0^{2k-1}} \\ \times \sum_{p \in \mathbb{N}} \sum_{s_i=1}^{k_0} \frac{t^P \sum_{i=1}^{2k} \epsilon_i P_i s_i'^2 / 2}{P^{2k}} \tag{2.47}$$

with the conditions  $\sum_{i=1}^{2k} \epsilon_i P_i = 0$ ,  $\sum_{i=1}^{2k} \epsilon_i P_i s_i$  is divisible by  $k_0$  ( $P_1, \dots, P_{2k}) = 1$  and  $\sum_{i=1}^{2k} \eta_i P_i \geq 0$ . As it was mentioned, for all  $P \in \mathbb{N}$ , we have

$$\left| \frac{1}{k_0^{2k-1}} \sum_{s_i=1}^{k_0} t^{P/2 \sum_{i=1}^{2k} \epsilon_i P_i s_i'^2} \right| < 1. \tag{2.48}$$

Note that for  $\alpha = 0$  ( $k_0 = 1$ ) this term is simply 1. Consequently we have the theorem

**Theorem 2.3.**  $\forall \alpha \in \mathbb{Q}, \forall k \in \mathbb{N}^* E[S_n^{2k}(\frac{\pi\beta}{4})] \underset{n \rightarrow \infty}{\simeq} C_k n^{2k-1}$  with  $C_1 = 1$  and  $C_k$  a constante.

This result suggests a unique normalization, for which the characteristic function of  $S_n(x, y; \alpha)$  once normalized will be independent of  $n$ , when  $n \rightarrow \infty$ :

$$\phi_{\frac{S_n}{n}}(t) \underset{n \rightarrow \infty}{\simeq} \sum_{k=0}^{\infty} \frac{(it)^{2k}}{(2k)!} C_k \frac{n^{2k-1}}{n^{2k}} = 1. \tag{2.49}$$

Consequently the probability distribution function  $F$  of the normalized random variable limit  $S = \lim_{n \rightarrow \infty} \frac{S_n(x, y; \alpha)}{n}$ , for  $\alpha \in \mathbb{Q}$ , is

$$F(S) = \delta \tag{2.50}$$

where  $\delta$  is the Dirac distribution. That follows in fact from the ergodic Birkhoff theorem, but the speed of the convergence to zero of  $\frac{S_n(x, y; \alpha)}{n}$  is very slow.

### 3. The case of irrational $\beta$ in terms of Gaussian sums

In this section we shall proceed with a different type of estimate of  $E(S_n^{2q}(\alpha))$  for  $\alpha \in \mathbb{R}^*$ . In particular, we emphasize the difference between the rational case of section 2 and the irrational case  $\alpha \in \mathbb{R} \setminus \mathbb{Q}$ . For this purpose, we shall establish a generalization of the duality formula for one-dimensional Gaussian sums, given in the celebrated work of Hardy and Littlewood [HL]. This problem consists in estimating the behaviour of the many variable sums, for large  $n$ :

$$\sum_{k_1=0}^{n-1} \dots \sum_{k_{2q}=0}^{n-1} \exp \left( i \frac{\alpha}{4} \sum_{i=2}^{2q} \sum_{j=2}^{2q} \frac{P_i P_j}{\sum_{i=2}^{2q} P_i} (k_i - k_j)^2 \right) \tag{3.1}$$

where  $P_i \in \mathbb{Z}, i = 2, \dots, 2q$ , are parameters (see equation (2.7)). The above sum bears over integer in indices of  $\mathbb{N}^{2q}$  formed by the intersection of the hypercube of total volume  $n^{2q}$  and the hyperplane defined by the equation

$$\sum_{i=2}^{2q} P_i (k_i - k_1) = 0. \tag{3.2}$$

This condition can be recast as an integral using the formulae

$$\int_0^1 e^{2\pi i x P} dx = 1 \quad \text{if } P = 0 \\ = 0 \quad \text{if } P \in \mathbb{N}^*. \tag{3.3}$$

Define now the quantity

$$\xi_n[x, \beta; P_2, \dots, P_{2q}] = \sum_{k_1=0}^{n-1} \dots \sum_{k_{2q}=0}^{n-1} \exp \left( i\pi\beta \sum_{i=2}^{2q} \sum_{j=2}^{2q} \frac{P_i P_j}{\sum_{i=2}^{2q} P_i} (k_i - k_j)^2 \right) \\ \times \exp \left( 2i\pi x \sum_{i=2}^{2q} P_i (k_i - k_1) \right) \tag{3.4}$$

Then, the  $2q$ th (see equation (2.7)) moment may be rewritten as

$$E \left[ S_n^{2q} \left( \frac{\pi\beta}{4} \right) \right] = (-1)^q \left( \frac{4}{\pi} \right)^q \sum_{P_i \in \mathbb{Z}} \frac{1}{\prod_{i=2}^{2q} P_i (\sum_{i=2}^{2q} P_i)} \int_0^1 \xi_n[x, \beta; P_2, \dots, P_{2q}] dx \tag{3.5}$$

As  $\xi_n[x, \beta; P_2, \dots, P_{2q}]$  are analogous to  $2q$  variables theta functions [M], we shall express formula (3.5) in terms of Gaussian sums  ${}^d\sigma_{A_{\vec{n}}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta})$ .

**Definition 5.** We define the  $d$ -dimensional Gaussian sums

$${}^d\sigma_{A_{\vec{n}}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \sum_{\vec{k} \in A_{\vec{n}}} \exp \left( i\pi^t \left( \vec{k} - \frac{\vec{a}}{2} \right) \Omega \left( \vec{k} - \frac{\vec{a}}{2} \right) \right) \exp \left( 2i\pi^t \left( \vec{k} - \frac{\vec{a}}{2} \right) \left( \vec{\theta} - \frac{\vec{b}}{2} \right) \right) \tag{3.6}$$

where  $\Omega$  is a  $d$ -square matrix with real coefficients:  $\Omega \in M_d(\mathbb{R}), \vec{\theta} \in \text{Vect}_d(\mathbb{R})$  such that  $\forall i = 1, \dots, d$ , the component  $\theta_i$  of  $\vec{\theta}$  is restricted to  $0 < \theta_i < 1$ . Here we denote by  ${}^t\vec{v}$  the transpose of a vector  $\vec{v}$ . Moreover  $\vec{a}$  and  $\vec{b}$  are elements of  $\text{Vect}_d(\{0, 1\})$ .  $A_{\vec{n}}$ , with  $\vec{n} = (n_1, \dots, n_d)$ , is the hyper-rectangle with integer sites in  $\mathbb{N}^d$  defined by  
 Any  $\vec{k} \in A_{\vec{n}}$  is such that  $\forall i = 1, \dots, d, 0 \leq k_i \leq n_i - 1, k_i \in \mathbb{N}$ .

With this notation

$$\xi_n[x, \beta; P_2, \dots, P_{2q}] = \sum_{k_1=0}^{n-1} \exp\left(-2i\pi x \left(\sum_{i=2}^{2q} P_i\right) k_1\right) \cdot {}^{2q-1}\sigma_{A_{\vec{n}}}^{[\vec{0}, \vec{0}]}(\Omega, \vec{\theta}) \tag{3.7}$$

where

- $A_{\vec{n}}$  defined by the  $(2q - 1)$ -vector  $\vec{n}$ :  $\vec{n} = (n - 1, \dots, n - 1)$
- the  $(2q - 1)$ -square matrix  $\Omega$ :

$$\Omega = \frac{\beta}{\sum P_i} \begin{bmatrix} P_2(\sum_{i=2}^{2q} P_i - P_2) & -P_2 P_3 & \dots & -P_2 P_{2q} \\ -P_2 P_3 & P_3(\sum_{i=2}^{2q} P_i - P_3) & \vdots & -P_3 P_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -P_2 P_{2q} & -P_3 P_{2q} & \dots & P_{2q}(\sum_{i=2}^{2q} P_i - P_{2q}) \end{bmatrix} \tag{3.8}$$

- the  $(2q - 1)$ -vector  $\theta$ :

$$\vec{\theta} = x \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_{2q} \end{bmatrix} \in \text{Vect}_{2q-1}(\mathbb{R}) \tag{3.9}$$

- $\vec{a} = \vec{b} = \vec{0}$ .

In order to compute the above sum, we shall first establish the duality formula:

**Theorem 3.1** (duality formula). *If  $\Omega \in M_d(]0, 1[)_{\text{sym,inv}}$  (the  $d$ -dimensional symmetric and invertible matrices with values in  $]0, 1[$ ),  $\vec{\theta} \in \text{Vect}_d(]0, 1[)$ , and  $(\vec{a}, \vec{b}) \in \text{Vect}_d(\{0, 1\})$ , then*

$${}^d\sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \left(\frac{i}{\sqrt{\det \Omega}}\right)^{\frac{d}{2}} e^{-i\pi' \vec{\theta} \Omega^{-1} \vec{\theta}} \times \left(\sum_{0 \leq i_1, \dots, i_d \leq d} \mathcal{O}(1)^{i_1 + \dots + i_d} \cdot {}^d\sigma_{\mathcal{P}_{i_1} \dots \mathcal{P}_{i_d}(\Omega \mathcal{D})}^{[\vec{b}, \vec{a}]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta})\right) \tag{3.10}$$

where  $\mathcal{D}$  is a domain of  $\mathbb{N}^d$  and  $\Omega(\mathcal{D})$  is the set of integers sites of  $\mathbb{N}^d$  contained in the image of  $\mathcal{D}$  by  $\Omega$  and  $\mathcal{P}_i$  is the projection on  $\{k_i = 0\}$ . Moreover we take the convention  $\mathcal{O}(1)^0 = 1$ .

This duality formula will be extended to  $\Omega \in M_d(\mathbb{R})_{\text{sym,inv}}$  ( $d$ -dimensional real symmetric and invertible matrices) and later we shall treat the case of singular  $\Omega$ , i.e.  $\det(\Omega) = 0$ .

### 3.1. Results of Hardy and Littlewood on one-dimensional Gaussian sums

Following [HL], let us introduce

$$\begin{aligned} C_n^2(x, \theta) &= \sum_{k=0}^{n-1} e^{i\pi x(k-1/2)^2} \cos(2k-1)\pi\theta \\ C_n^3(x, \theta) &= \sum_{k=0}^{n-1} e^{i\pi x k^2} \cos 2k\pi\theta \\ C_n^4(x, \theta) &= \sum_{k=0}^{n-1} (-1)^k e^{i\pi x k^2} \cos 2k\pi\theta. \end{aligned} \tag{3.11}$$

Then, the duality formula for the Gaussian sum  $C_n^3(x, \theta)$  (see equation (3.17)) can be obtained by the application of the residue theorem in the formula [HL]

$$C_n^3(x, \theta) = \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} C_{nx}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + i \int_0^\infty \frac{e^{-\pi x(n^2-t^2)}}{\sinh \pi t} Q_n(\theta, x, t) dt \tag{3.12}$$

with  $0 < x < 1$  and  $0 < \theta < 1$ . The function  $Q_n(\theta, x, t)$  is given by

$$Q_n(\theta, x, n) = \cos(2\pi n\theta) \cosh(2\pi\theta t) \sinh((2nx - 2k - 1)\pi t) + i \sin(2\pi n\theta) \sinh(2\pi\theta t) \cosh((2nx - 2k - 1)\pi t). \tag{3.13}$$

Observe that the integral in this equation may be reduced to integrals of the type

$$\int_0^\infty e^{i\pi x(n^2-t^2)} e^{2\pi\theta t} \frac{\sinh(\alpha\pi t)}{\sinh(\pi t)} dt. \tag{3.14}$$

From the theorem on intermediate values we have  $\exists T$  bounded, such that  $\forall \theta \in ]0, 1[$ , the previous integral is

$$\mathcal{O}(1) e^{-\pi n^2 x} \int_0^{T(\theta)} e^{i\pi x t^2} e^{2\pi\theta t} dt \tag{3.15}$$

where  $\mathcal{O}(1)$  means a bounded function with respect to  $\theta$ . This integral depends on  $\theta$  but remains bounded. The validity of this theorem is guaranteed since  $\frac{\sinh(\alpha\pi t)}{\sinh(\pi t)}$  is a positive decreasing function of  $\theta$  on  $\mathbb{R}^+$ . Its value is

$$\mathcal{O}(1) \sqrt{\frac{i}{x}} e^{-\pi \frac{\theta^2}{x}} \left( \operatorname{erf}\left(T\sqrt{i\alpha} - \frac{b}{\sqrt{i\alpha}}\right) - \operatorname{erf}\left(\frac{b}{\sqrt{i\alpha}}\right) \right) \tag{3.16}$$

where the function erf (the error function) is defined as  $\operatorname{erf}(z) \mapsto \int_0^z e^{-\pi v^2} dv$ . This leads to a more precise form of the result of Hardy and Littelwood. A similar proof can be performed for  $C_n^2$  and  $C_n^4$ . In fact we have obtained the proposition:

**Proposition 6.** *We have the following duality formulae:  $\forall 0 < x < 1$  and  $\forall 0 < \theta < 1$*

$$\begin{aligned} C_n^2(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{nx}^4\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \\ C_n^3(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{nx}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \\ C_n^4(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{nx}^2\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right). \end{aligned} \tag{3.17}$$

In their work, Hardy and Littlewood have stated that analogous sums with sine instead cosine also obey duality formulae which can be established in a similar way. However as already seen, the parity of the cosine function does play an important role. When the cosine is replaced by the sine, there appears Fresnel functions which need additional treatment. We offer here an alternative proof. Define

$$C_{q,q+r}^3(x, \theta) = \sum_{k=q+1}^{q+r} e^{i\pi x k^2} \cos(2k\pi\theta) = C_{q+r}^3(x, \theta) - C_q^3(x, \theta). \tag{3.18}$$

Applying the duality formula on  $C_q^3(x, \theta)$  we find

$$C_{q,q+r}^3(x, \theta) = \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{nq,n(q+r)}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right). \tag{3.19}$$



But we also have

$$C_{q,q+r}^3(x, \theta) = \sum_{k=q+1}^{q+r} e^{i\pi x k^2} \cos(2\pi k\theta) = \sum_{k=1}^{q+r} e^{i\pi x(k+q)^2} \cos 2\pi(k+q)\theta \quad (3.20)$$

after the expansion of the cosine, we obtain

$$\begin{aligned} C_{q,q+r}^3(x, \theta) &= \frac{e^{-\pi x q^2}}{2} \sum_{k=1}^r e^{i\pi x k^2} (\cos 2\pi \theta q [\cos 2\pi(xq + \theta)k + \cos 2\pi(xq - \theta)k \\ &\quad + i(\sin 2\pi(xq + \theta)k + \sin 2\pi(xq - \theta)k)] + \sin 2\pi \theta q [\sin 2\pi(xq + \theta)k \\ &\quad - \sin 2\pi(xq - \theta)k - i(\sin 2\pi(xq + \theta)k + \sin 2\pi(xq - \theta)k)]). \end{aligned} \quad (3.21)$$

Assume now  $x \in \mathbb{Q}$ , such that  $x = a/b$ , where  $a$  and  $b$  are prime numbers. We choose now  $e_1$  as a multiple of  $b$  and let  $e(xq)$  label the integer part of  $xq$ :

$$\begin{aligned} C_{q,q+r}^3(x, \theta) &= (-1)^{qe(xq)} \\ &\quad \times \left[ \cos 2\pi q\theta \sum_{k=0}^{r-1} e^{i\pi x k^2} \cos 2\pi \theta k - \sin 2\pi q\theta \sum_{k=0}^{r-1} e^{i\pi x k^2} \sin 2\pi \theta k \right]. \end{aligned} \quad (3.22)$$

Using the duality formula we obtain

$$\begin{aligned} C_{q,q+r}^3(x, \theta) &= (-1)^{qe(xq)} \cos 2\pi q\theta \sqrt{\frac{1}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{xr}^3 \left( -\frac{1}{x}, \frac{\theta}{x} \right) + \mathcal{O}(1) \right) \\ &\quad - (-1)^{qe(xq)} \sin 2\pi q\theta \sum_{k=0}^{r-1} e^{i\pi x k^2} \sin 2\pi \theta k. \end{aligned} \quad (3.23)$$

Moreover we have

$$\begin{aligned} &\sqrt{\frac{1}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{xr}^3 \left( -\frac{1}{x}, \frac{\theta}{x} \right) + \mathcal{O}(1) \right) \\ &= \sqrt{\frac{1}{x}} e^{-i\pi \frac{\theta^2}{x}} \sum_{k=0}^{e(xr)-1} \sqrt{\frac{1}{x}} e^{-i\pi \frac{\theta^2}{x}} (-1)^{qe(xq)} e^{-i\pi \frac{(k+e(xq))^2}{x}} \cos 2\pi \frac{k+e(xq)}{x} \theta \\ &\quad \times \left[ \cos 2\pi \frac{e(xq)}{x} \theta \sum_{k=0}^{e(xr)-1} e^{-i\pi \frac{k^2}{x}} \cos 2\pi \frac{k}{x} \theta \right. \\ &\quad \left. - \sin 2\pi \frac{e(xq)}{x} \theta \sum_{k=0}^{e(xr)-1} e^{-i\pi \frac{k^2}{x}} \sin 2\pi \frac{k}{x} \theta + \mathcal{O}(1) \right]. \end{aligned} \quad (3.24)$$

This leads to the equality

$$\begin{aligned} &\cos 2\pi \theta q \left( C_{xr}^3 \left( -\frac{1}{x}, \frac{\theta}{x} \right) + \mathcal{O}(1) \right) - \sqrt{\frac{x}{1}} \sin 2\pi q\theta \sum_{k=0}^{e(xr)-1} e^{i\pi \frac{\theta^2 - k^2}{x}} \sin 2\pi \frac{k}{x} \theta \\ &= C_{xr}^3 \left( -\frac{1}{x}, \frac{\theta}{x} \right) \cos 2\pi \frac{e(xq)}{x} \theta - S_{xr}^3 \left( -\frac{1}{x}, \frac{\theta}{x} \right) \sin 2\pi \frac{e(xq)}{x} \theta + \mathcal{O}(1) \end{aligned} \quad (3.25)$$

when  $q$  is given as a multiple of  $b$ , we simply get

$$S_r^3(x, \theta) = \sqrt{\frac{1}{x}} e^{-i\pi \frac{\theta^2}{x}} \left[ S_{xr}^3 \left( -\frac{1}{x}, \frac{\theta}{x} \right) + \mathcal{O}(1) \right] \quad (3.26)$$

$\forall \theta$  such that  $2\pi q\theta \neq \pi k$  or  $\theta \not\equiv \frac{k}{2q} \pmod{1}$ . Let us now call

$$\zeta_q = \left\{ \theta \in ]0, 1[ \mid \exists k \in \mathbb{N} \text{ such that } \theta \equiv \frac{k}{2q} \pmod{1} \right\}. \tag{3.27}$$

But  $\forall \theta, \exists q'$  a multiple of  $b$  such that  $\theta \notin \zeta_{q'}$ .

Hence,  $\forall \theta \in ]0, 1[$  and  $\forall x \in \mathbb{Q}$ , we obtain the same duality formula. Finally  $S_r(x, \theta)$  is a continuous function and bicontinuous since it is a finite sum of bicontinuous functions. As  $\mathbb{Q}$  is dense  $\mathbb{R}$ ,  $\forall x \in ]0, 1[$ ,  $\exists x_n \in \mathbb{Q} \setminus ]0, 1[$ ,  $n \in \mathbb{N}$  such that  $\lim_{n \rightarrow \infty} x_n = x$  and by continuity we obtain the limits

$$\begin{aligned} \lim_{n \rightarrow \infty} S_r^3(x_n, \theta) &= S_r^3(x, \theta) \\ \lim_{n \rightarrow \infty} S_{xr}^3\left(\frac{-1}{x_n}, \frac{\theta}{x_n}\right) &= S_{xr}^3\left(\frac{-1}{x}, \frac{\theta}{x}\right). \end{aligned} \tag{3.28}$$

Doing the same proof for  $S_n^2(x, \theta)$  and  $S_n^4(x, \theta)$  we get the following duality formulae:

$$\begin{aligned} S_n^2(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( S_{nx}^4\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \\ S_n^3(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( S_{nx}^3\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \\ S_n^4(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( S_{nx}^2\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \end{aligned} \tag{3.29}$$

These results are used to show that for the following Gaussian sums,

$${}^1\sigma_n^{[a,b]}(x, \theta) = \sum_{k=0}^{n-1} e^{-i\pi x(k-a/2)^2} e^{2i\pi(k-a/2)(\theta-b/2)} \tag{3.30}$$

with  $a = 1, 0$  and  $b = 0, 1$ , we have  $\forall x \in ]0, 1[$  and  $\theta \in ]0, 1[$ :

$${}^1\sigma_n^{[a,b]}(x, \theta) = \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( {}^1\sigma_{nx}^{[a,b]}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right). \tag{3.31}$$

We summarize these results in the following:

**Theorem 3.2.**  $\forall x \in ]0, 1[$ ,  $\forall \theta \in ]0, 1[$ ,  $\forall a, b = 0, 1$ :

$$\begin{aligned} {}^1\sigma_n^{[a,b]}(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( {}^1\sigma_{nx}^{[b,a]}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \\ C_n^{[a,b]}(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( C_{nx}^{[b,a]}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \\ S_n^{[a,b]}(x, \theta) &= \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( S_{nx}^{[b,a]}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right). \end{aligned} \tag{3.32}$$

Upon application of the duality  $\forall x \in ]0, 1[$ ,  $\forall \theta \in ]0, 1[$ ,  $\forall a, b = 0, 1$ , we have

$$\begin{aligned} {}^1\sigma_n^{[a,b]}(-x, \theta) &= [{}^1\sigma_n^{[a,b]}(x, \theta)]^* \\ &= \left[ \sqrt{\frac{i}{x}} e^{-i\pi \frac{\theta^2}{x}} \left( {}^1\sigma_{nx}^{[a,b]}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right) \right]^* \\ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{-x}} e^{i\pi \frac{\theta^2}{x}} \left[ {}^1\sigma_{nx}^{[b,a]}\left(-\frac{1}{x}, \frac{\theta}{x}\right) + \mathcal{O}(1) \right]^* \\ &= \frac{e^{-i\frac{\pi}{4}}}{\sqrt{-x}} e^{i\pi \frac{\theta^2}{-x}} \left[ {}^1\sigma_{-nx}^{[b,a]}\left(-\frac{1}{-x}, \frac{\theta}{-x}\right) + \mathcal{O}(1) \right]^*. \end{aligned} \tag{3.33}$$

We extend the validity of the duality formula to  $x \in \mathbb{R} \setminus \mathbb{N}$ . Using the convention  $\sqrt{-1} = i$ , the identity formula can be extended for  $-1 < x < 1$  and  $x \neq 0$ . All this remains valid for  $C_n$  and  $S_n$ , since we have the identity

$${}^1\sigma_n^{[a,b]}(x, \theta) = C_n^{[a,b]}(x, \theta) + iS_n^{[a,b]}(x, \theta) \tag{3.34}$$

Finally we also have the formula

$${}^1\sigma_n^{[a,b]}(x + 1, \theta) = \sqrt{i} {}^1\sigma_n^{[a,b']}(x, \theta) \tag{3.35}$$

with  $b' \equiv b + a \pmod{1}$ , it may be used to extend the validity domain of the formula into  $x \in \mathbb{R} \setminus \mathbb{Z}$ . If  $x \in \mathbb{Z}$ , the series reduces to a geometric series. The same results hold for  $C_n$  and  $S_n$ .

**Remark 7.** If  $x \in \mathbb{Q}$ , there exists a discrete duality formula (Shaar’s formula). This relation is actually the same as our formulae without the term  $\mathcal{O}(1)$ . The formulae are used in successive iterations until the summations are limited to a small number. The domain of summation is reduced at each step. However the result depends on its expansion as a continuous fraction (see [HL]). This last problem is closely related to the Gauss transform  $T(x) \equiv \frac{1}{x} \pmod{1}$  for which there exist infinite generating partitions. So for example if  $x$  is irrational, but admits a decomposition into continuous bounded fractions, we have

$${}^1\sigma_n^{[a,b]}(x, \theta) \simeq \mathcal{O}(\sqrt{n}) \tag{3.36}$$

a homogenous function in  $\theta$ . But if  $x \in \mathbb{Q}$

$${}^1\sigma_n^{[a,b]}(x, \theta) \simeq \mathcal{O}(n). \tag{3.37}$$

#### 4. Two-dimensional Gaussians sums

##### 4.1. Two-dimensional Gaussian sums with non-degenerate matrix $\Omega$

Once this case treated, one can extend the results to Gaussian sums of arbitrary dimensions, provided some delicate points are worked. Consider the Gaussian sum in two dimensions:

$${}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta}) = \sum_{k_1=0}^{q_1-1} \sum_{k_2=0}^{q_2-1} e^{i\pi(\omega_1 k_1^2 + \omega_2 k_2^2 + 2\omega k_1 k_2)} e^{2i\pi(k_1 \theta_1 + k_2 \theta_2)} \tag{4.1}$$

where  $\vec{q} = (q_1, q_2)$  and  $\Omega \in M(\{0, 1\}_{\text{sym}})$  with

$$\Omega = \begin{bmatrix} \omega_1 & \omega \\ \omega & \omega_2 \end{bmatrix}. \tag{4.2}$$

$A_{\vec{q}}$  labels integer sites of rectangles of dimensions  $q_1 \times q_2$ . We first assume that  $0 < \omega q_1 + \theta_1 < 1$  and  $0 < \omega q_2 + \theta_2 < 1$ . As  $q_1$  and  $q_2$  are to be large numbers, one may call these matrices *weak coupling* matrices. If  $\omega = 0$ , this Gaussian sum goes back to a product of one-dimensional Gaussian sums. In general

$${}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta}) = \sum_{k_1=0}^{q_1-1} e^{i\pi(\omega_1 k_1^2 + 2\theta_1 k_1)} \sum_{k_2=0}^{q_2-1} e^{i\pi(\omega_2 k_2^2 + 2\omega k_1 k_2 + 2\theta_2 k_2)}. \tag{4.3}$$

We shall establish the duality formula successively with respect to  $(k_1, k_2)$ . If the duality transformation is performed on the  $k_2$  sum one gets

$$\begin{aligned}
 {}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta}) &= \sum_{k_1=0}^{q_1-1} \exp(i\pi (w_1 k_1^2 + 2\theta_1 k_1)) \exp\left(-i\pi \frac{(\omega k_1 + \theta_2)^2}{\omega_2}\right) \\
 &\quad \times \sqrt{\frac{i}{\omega_2}} \left( \sum_{k_2=0}^{\omega_2(q_2-1)} \exp\left(-i\pi \frac{k_2^2}{\omega_2}\right) \exp\left(2i\pi \frac{(\omega k_1 + \theta_2)^2}{\omega_2}\right) + \mathcal{O}(1) \right) \\
 &= \sqrt{\frac{i}{\omega_2}} \exp\left(-i\pi \frac{\theta_2^2}{\omega_2}\right) \left( \sum_{k_2=0}^{\omega_2(q_2-1)} \exp\left(-i\pi \frac{k_2^2}{\omega_2}\right) \exp\left(2i\pi \frac{\theta_2}{\omega_2} k_2\right) \right. \\
 &\quad \times \exp\left(2i\pi \left(\theta_1 - \theta_2 \frac{\omega}{\omega_2} + \frac{\omega}{\omega_2} k_2\right) k_1\right) \sum_{k_1=0}^{q_1-1} \exp\left(i\pi \frac{\omega_1 \omega_2 - \omega^2}{\omega_2} k_1^2\right) \Big) \\
 &\quad + \mathcal{O}(1) \sqrt{\frac{i}{\omega_2}} \exp\left(-i\pi \frac{\theta_2^2}{\omega_2}\right) \sum_{k_1=0}^{q_1-1} \exp\left(i\pi \frac{\omega_1 \omega_2 - \omega^2}{\omega_2} k_1^2\right) \\
 &\quad \times \exp\left(2i\pi \left(\theta_1 - \frac{\omega}{\omega_2} \theta_2\right) k_1\right) \tag{4.4}
 \end{aligned}$$

when  $\Omega$  is an invertible matrix ( $\det(\Omega) \neq 0$ ), we can further apply the duality transformation on the  $k_1$  sum and obtain

$$\begin{aligned}
 {}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta}) &= \sqrt{\frac{i}{\omega_2}} \exp\left(-i\pi \frac{\theta_2^2}{\omega_2}\right) \left( \sum_{k_2=0}^{\omega_2 q_2 - 1} \exp\left(-i\pi \left(\frac{k_2^2}{\omega_2} + 2\frac{\theta_2}{\omega_2} k_2\right)\right) \right. \\
 &\quad \times \exp\left(-i\pi \frac{\omega_2 \theta_2 - \omega \theta_2 + \omega k_2}{\omega_2(\omega_1 \omega_2 - \omega^2)} k_1\right) \Big) \\
 &\quad \times \left( \sum_{k_1=0}^{(\omega_1 - \frac{\omega^2}{\omega_2}) q_1 - 1} \exp\left(-i\pi \frac{\omega_2}{\omega_1 \omega_2 - \omega^2} k_1^2\right) \exp\left(2i\pi \frac{\omega_2 \theta_1 - \omega \theta_2 + \omega k_2}{\omega_2(\omega_1 \omega_2 - \omega^2)} k_1\right) \right) \\
 &\quad + \mathcal{O}(1) \sqrt{\frac{i}{\omega_2}} \exp\left(-i\pi \frac{\theta_2^2}{\omega_2}\right) \exp\left(-i\pi \frac{\omega_2}{\omega_1 \omega_2 - \omega^2} \left(\theta_2 - \frac{\omega}{\omega_2} \theta_1\right)\right) \\
 &\quad \times \left( \sum_{k_2=0}^{(\omega_1 - \frac{\omega^2}{\omega_2}) q_2 - 1} \exp\left(-i\pi \frac{\omega_2}{\omega_1 \omega_2 - \omega^2} k_2^2\right) \exp\left(2i\pi \frac{\omega_2 \theta_2 - \omega \theta_2}{\omega_2(\omega_1 \omega_2 - \omega^2)} k_2\right) + \mathcal{O}(1) \right). \tag{4.5}
 \end{aligned}$$

Now taking into account the assumptions on  $\omega$ , the summation over the variable  $k_1$  from 0 to  $(\omega_1 - \frac{\omega^2}{\omega_2}) q_1 - 1$  is the same as the summation over  $k_1 = 0, \dots, \omega_1 q_1 - 1$ . This fact also occurs for the  $k_2$  such that the above expression becomes

$$\begin{aligned}
 {}^2\sigma_{A_{(q_1, q_2)}}^{[0,0]}(\Omega, \vec{\theta}) &= \frac{i e^{-i\pi^t \vec{\theta} \Omega^{-1} \vec{\theta}}}{\sqrt{\det(\Omega)}} \left[ {}^2\sigma_{A_{(\omega_1 q_1, \omega_2 q_2)}}^{[0,0]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}) \right. \\
 &\quad + \mathcal{O}(1) \left( {}^2\sigma_{\mathcal{P}_1[A_{(\omega_1 q_1, \omega_2 q_2)}]}^{[0,0]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}) \right. \\
 &\quad \left. \left. + {}^2\sigma_{\mathcal{P}_2[A_{(\omega_1 q_1, \omega_2 q_2)}]}^{[0,0]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}) + 1 \right) \right] \tag{4.6}
 \end{aligned}$$

where  $\mathcal{P}_1$  (resp.  $\mathcal{P}_2$ ) is the projection onto the plane  $\{k_1 = 0\}$  (resp.  $\{k_2 = 0\}$ ). We also note that the duality transformation essentially replaces sums over  $k$  by other sums over  $k$  with an

error of order 1. For example  $\sum_{k_1, k_2}$  is replaced by  $(\sum_{k_1} + \mathcal{O}(1)) (\sum_{k_2} + \mathcal{O}(1)) \simeq \sum_{k_1, k_2} + \mathcal{O}(1) \sum_{k_1} + \mathcal{O}(1) \sum_{k_2} + \mathcal{O}^2(1)$ . Moreover, there is a simplification since

$${}^2\sigma_{\mathcal{P}_1[A(\omega q_1, \omega_2 q_2)]}^{[0,0]}(-\Omega^{-1}, \Omega^{-1}\vec{\theta}) = {}^1\sigma_{A(\omega_2 q_2)}^{[0]} \left( -\frac{\omega_1}{\omega_1 \omega_2 - \omega^2}, \frac{\omega_1 \theta_2 - \omega \theta_1}{\omega_1 \omega_2 - \omega^2} \right). \tag{4.7}$$

For sufficiently small  $\omega$ , integer sites such that  $0 \leq k_1 \leq \omega_1 q_1$  and  $0 \leq k_2 \leq \omega_2 q_2$  are those with  $\vec{k} = (k_1, k_2) \in \Omega(A_{\vec{q}})$ . Thus

$${}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta}) = \frac{i e^{-i\pi^t \vec{\theta} \Omega^{-1} \vec{\theta}}}{\sqrt{\det(\Omega)}} \left[ {}^2\sigma_{\Omega A_{\vec{q}}}^{[0,0]}(-\Omega^{-1}, \Omega^{-1}\vec{\theta}) + \mathcal{O}(1) \cdot {}^2\sigma_{\mathcal{P}_1[\Omega A_{\vec{q}}]}^{[0,0]}(-\Omega^{-1}, \Omega^{-1}\vec{\theta}) + \mathcal{O}(1) \cdot {}^2\sigma_{\mathcal{P}_2[\Omega A_{\vec{q}}]}^{[0,0]}(-\Omega^{-1}, \Omega^{-1}\vec{\theta}) + \mathcal{O}(1) \right]. \tag{4.8}$$

Now for  $\vec{a}$  and  $\vec{b}$  nonzero (i.e.  $a_i = \{0, 1\}$ ;  $b_i = \{0, 1\}$ ,  $i = 1, 2$ ), we have in a similar manner the lemma

**Lemma 8.**  $\forall \Omega \in M_2(\mathbb{R})_{\text{sym}}$  an invertible matrix and with  $\omega$  small enough:

$${}^2\sigma_{A_{\vec{q}}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \frac{i e^{-i\pi^t \vec{\theta} \Omega^{-1} \vec{\theta}}}{\sqrt{\det(\Omega)}} \left[ {}^2\sigma_{\Omega A_{\vec{q}}}^{[\vec{b}, \vec{a}]}(-\Omega^{-1} \Omega^{-1}, \vec{\theta}) + \mathcal{O}(1) \cdot {}^2\sigma_{\mathcal{P}_1[\Omega A_{\vec{q}}]}^{[\vec{b}, \vec{a}]}(-\Omega^{-1} \Omega^{-1}, \vec{\theta}) + \mathcal{O}(1) \cdot {}^2\sigma_{\mathcal{P}_2[\Omega A_{\vec{q}}]}^{[\vec{b}, \vec{a}]}(-\Omega^{-1} \Omega^{-1}, \vec{\theta}) + \mathcal{O}(1) \right] \tag{4.9}$$

where

$${}^2\sigma_{A_{\vec{q}}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \sum_{\vec{k} \in A_{\vec{q}}} \exp \left( i\pi^t \left( \vec{k} - \frac{\vec{a}}{2} \right) \Omega \left( \vec{k} - \frac{\vec{a}}{2} \right) \right) \exp \left( 2\pi i^t \left( \vec{\theta} - \frac{\vec{b}}{2} \right) \left( \vec{k} - \frac{\vec{a}}{2} \right) \right).$$

4.2. Two-dimensional Gaussian sums with a degenerate matrix  $\Omega$

In the expression of  ${}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta})$  see equation (4.1) we perform the duality transformation on  $k_2$  sums. This operation yields

$${}^2\sigma_{A_{\vec{q}}}^{[0,0]}(\Omega, \vec{\theta}) = \sqrt{\frac{i}{\omega_2}} \exp \left( -i\pi \frac{\theta_2^2}{\Omega_2} \right) \exp \left( i\pi (q - 1) \left( \theta_1 - \frac{\omega}{\omega_2} \theta_2 \right) \right) \times \left( \sum_{k_2=0}^{\omega_2 q_2 - 1} \exp \left( -i\pi \frac{k_2^2}{\omega_2} \right) \exp \left( 2i\pi \left( \frac{\theta_2}{\omega_2} + \frac{(q-1)\omega}{2\omega_2} \right) k_2 \right) \times \frac{\sin \pi q_1 \left( \theta_1 - \frac{\omega}{\omega_2} (k_2 - 1) \right)}{\sin \pi \left( \theta_1 - \frac{\omega}{\omega_2} (k_2 - 1) \right)} + \mathcal{O}(1) \frac{\sin \pi q_1 \left( \theta_1 - \frac{\omega}{\omega_2} \right)}{\sin \pi \left( \theta_1 - \frac{\omega}{\omega_2} \right)} \right). \tag{4.10}$$

We observe that for  $\Omega$  non-invertible, the general form of the sum  $e^{i\pi \text{quadratic form}}$   $e^{i\pi \text{linear form}}$  is not preserved under this duality transformation. Moreover the symmetry  $k_1 \leftrightarrow k_2$  seems to be apparently broken since its expression depends on the variable which is to be transformed first by duality. This type of formula will be used later on, although with an appropriate form. In the present case, it is sufficient to perform successive dualities uniquely on the variable  $k_2$  until the summation on  $k_2$  is equal to  $\mathcal{O}(1)$ . Then perform at last the geometric summation over  $k_1$ . We shall now remove the restriction on small  $\omega$  and consider the general case. Let us give the following definition:

**Definition 9.** An invertible symmetric matrix is called non-diagonal integer, when there are no integer entries outside its diagonal. This set of such matrices is denoted by  $M_{\mathbb{Z}}^*(\mathbb{R})_{\text{inv, sym}}$ .

Let

$$f_{[\Omega, \vec{\theta}]}(\vec{k} + \vec{n}) = e^{i\pi'(\vec{k} + \vec{n})\Omega(\vec{k} + \vec{n})} e^{2i\pi'(\vec{k} + \vec{n})\vec{\theta}} \tag{4.11}$$

with  $\vec{\theta}' \equiv \vec{\theta} + \Omega\vec{n} \pmod{1}$ , we have

$$\begin{aligned} f_{[\Omega, \vec{\theta}]}(\vec{k} + \vec{n}) &= e^{2\pi i' \vec{k} \Omega \vec{n}} f_{[\Omega, \vec{\theta}]}(\vec{k}) f_{[\Omega, \vec{\theta}]}(\vec{n}) \\ &= f_{[\Omega, \vec{\theta}]}(\vec{n}) f_{[\Omega, \vec{\theta}]}(\vec{k}). \end{aligned} \tag{4.12}$$

So for  $\theta_1$  and  $\theta_2$  given, let  $(q'_1, q'_2)$  be such that the duality formula is valid, we cut the domain  $\{0 \leq k_1 < q_1, 0 \leq k_2 < q_2\}$  into small rectangles of size  $q'_1 \times q'_2$ . Let us consider the set

$$B(m_1, m_2) = \{(k_1, k_2) \in \mathbb{N}^2 \mid m_1 q'_1 \leq k_1 < (m_1 + 1)q'_1 \text{ and } m_2 q'_2 \leq k_2 < (m_2 + 1)q'_2\}. \tag{4.13}$$

Necessarily  $m_1 \leq e(q_1/q'_1)$  and  $m_2 \leq e(q_2/q'_2)$ ; here  $e(q/q')$  represents the integer part of  $q/q'$ . It is sufficient to verify that the duality formula is proved by shicking together the  $B(m_1, m_2)$ . Call  $\vec{m} = (m_1, m_2)$ . Then

$$\sum_{\vec{k} \in B(\vec{m})} f_{[\Omega, \vec{\theta}]}(\vec{k}) = \sum_{\vec{k} \in B(\vec{0})} f_{[\Omega, \vec{\theta}]}(\vec{k} + \vec{m}) = f_{[\Omega, \vec{\theta}]}(\vec{m}) \sum_{\vec{k} \in \Omega B(\vec{0})} f_{[\Omega, \vec{\theta}']}(\vec{k}) \tag{4.14}$$

with  $\vec{\theta}'' = \vec{\theta} + \Omega\vec{m} - \vec{N}$ , where  $\vec{N}$  is a vector with integer coordinates such that  $\forall i, \theta''_i = \theta_i + (\Omega\vec{m})_i - N_i \in ]0, 1[$ . Now we perform the duality transformation

$$\begin{aligned} \sum_{\vec{k} \in B(\vec{m})} f_{[\Omega, \vec{\theta}]}(\vec{k}) &= f_{[\Omega, \vec{\theta}]}(\vec{m}) i \frac{e^{-i\pi' \vec{\theta}'' \Omega^{-1} \vec{\theta}''}}{\det(\Omega)} \left[ \sum_{\vec{k} \in \Omega B(\vec{0})} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}']}(\vec{k}) \right. \\ &\quad \left. + \mathcal{O}(1) \left( \sum_{\vec{k} \in \mathcal{P}_1 \Omega B(\vec{0})} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}']}(\vec{k}) + \sum_{\vec{k} \in \mathcal{P}_2 \Omega B(\vec{0})} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}']}(\vec{k}) + 1 \right) \right] \end{aligned} \tag{4.15}$$

with

$$e^{i\pi' \vec{\theta}'' \Omega \vec{\theta}''} = e^{i\pi' \vec{\theta} \Omega^{-1} \vec{\theta}} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{N}) f_{[-\Omega, -\vec{\theta}]}(\vec{m}) \tag{4.16}$$

since  ${}^t \vec{N} \cdot \vec{m} \in \mathbb{N}$ . Finally

$$\begin{aligned} \sum_{\vec{k} \in B(\vec{m})} f_{[\Omega, \vec{\theta}]}(\vec{k}) &= i \frac{e^{-i\pi' \vec{\theta} \Omega^{-1} \vec{\theta}}}{\sqrt{\det(\Omega)}} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{N}) \left[ \sum_{\substack{\vec{k} \in \\ \Omega B(\vec{0})}} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta} - \Omega^{-1} \vec{N}]}(\vec{k}) \right. \\ &\quad \left. + \mathcal{O}(1) \left( \sum_{\vec{k} \in \mathcal{P}_\infty \Omega B(\vec{0})} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta} - \Omega^{-1} \vec{N}]}(\vec{k}) + \sum_{\vec{k} \in \mathcal{P}_2 \Omega B(\vec{0})} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta} - \Omega^{-1} \vec{N}]}(\vec{k}) + 1 \right) \right] \end{aligned} \tag{4.17}$$

or alternatively

$$\sum_{\vec{k} \in B(\vec{m})} f_{[\Omega, \vec{\theta}]}(\vec{k}) = i \frac{e^{-i\pi' \vec{\theta} \Omega \vec{\theta}}}{\det(\Omega)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{N}) \left[ \sum_{\substack{(\vec{k}-\vec{N}) \in \\ \Omega B(\vec{0})}} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{N}) \right. \\ \left. + \mathcal{O}(1) \left( \sum_{\substack{(\vec{k}-\vec{N}) \in \\ \mathcal{P}_1 \Omega B(\vec{0})}} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) + \sum_{\substack{(\vec{k}-\vec{N}) \in \\ \mathcal{P}_2 \Omega B(\vec{0})}} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) + 1 \right) \right]. \quad (4.18)$$

Here  $\vec{N}$  is an integer vector such that  $\forall i = 1, 2, |N_i - (\Omega \vec{m})_i| < 1$ . Thus replacing  $\vec{N}$  by  $\Omega \vec{m}$  we make an error in the double sum  $\sum_{k_1, k_2}$  of the order  $\mathcal{O}(1) \sum_{k_1}$  and  $\mathcal{O}(1) \sum_{k_2}$  and an error with respect to the  $\sum_{k_1}$  (resp.  $\sum_{k_2}$ ) of order  $\mathcal{O}(1)$ . The index  $(\vec{k} - \Omega \vec{m}) \in \Omega B(\vec{0})$  is equivalent to  $\vec{k} \in \Omega(\vec{m} - B(\vec{0}))$  or also  $\vec{k} \in \Omega B(\vec{m})$  by linearity. For the same reasons we have, for  $i = 1, 2, (\vec{k} - \Omega \vec{m}) \in \mathcal{P}(\Omega B(\vec{0}))$  is equivalent to  $\vec{k} \in \mathcal{P}(\Omega B(\vec{m}))$  and we have

$$\sum_{k_1=0}^{q_1-1} \sum_{k_2=0}^{q_2-1} f_{[\Omega, \vec{\theta}]}(\vec{k}) = \sum_{r=0}^{e(q_1/q'_1)} \sum_{s=0}^{e(q_2/q'_2)} \sum_{\vec{k} \in B(r,s)} f_{[\Omega, \vec{\theta}]}(\vec{k}) i \frac{e^{-i\pi' \vec{\theta} \Omega \vec{\theta}}}{\sqrt{\det(\Omega)}} \sum_{r=0}^{e(q_1/q'_1)} \sum_{s=0}^{e(q_2/q'_2)} \\ = \left[ \sum_{\vec{k} \in \Omega B(r,s)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) \right. \\ \left. + \mathcal{O}(1) \left( \sum_{\vec{k} \in \mathcal{P}_1 \Omega B(r,s)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) + \sum_{\vec{k} \in \mathcal{P}_2 \Omega B(r,s)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) + 1 \right) \right] \\ = i \frac{e^{-i\pi' \vec{\theta} \Omega \vec{\theta}}}{\det(\Omega)} \left[ \sum_{\vec{k} \in \Omega A(q_1, q_2)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) \right. \\ \left. + \mathcal{O}(1) \left( \sum_{\vec{k} \in \mathcal{P}_1 \Omega A(q_1, q_2)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) + \sum_{\vec{k} \in \mathcal{P}_2 \Omega A(q_1, q_2)} f_{[-\Omega^{-1}, \Omega^{-1} \vec{\theta}]}(\vec{k}) + 1 \right) \right]. \quad (4.19)$$

Finally the same calculation shows that for all bounded domains  $\mathcal{D} \in \mathbb{N}^2$ , we have the theorem

**Theorem 4.1** (Duality). *Let  $\Omega \in M_2^*(]0, 1[)_{\text{inv}}$ ,  $\mathcal{D}$  the bounded domain in  $\mathbb{N}^2$  we have*

$${}^2\sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = i \frac{e^{-i\pi' \vec{\theta} \Omega \vec{\theta}}}{\det(\Omega)} \left[ {}^2\sigma_{\Omega \mathcal{D}}^{[\vec{a}, \vec{b}]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}) + \mathcal{O}(1) \cdot {}^2\sigma_{\mathcal{P}_{\infty}(\Omega \mathcal{D})}^{[\vec{a}, \vec{b}]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}) \right. \\ \left. + \mathcal{O}(1) \cdot {}^2\sigma_{\mathcal{P}_{\infty}(\Omega \mathcal{D})}^{[\vec{a}, \vec{b}]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}) + \mathcal{O}(1) \right]. \quad (4.20)$$

This formula can be extended to the set of matrices  $\Omega \in M_2^*(\mathbb{R})_{\text{inv}}$ . This result is given in the next section where the general case with  $d$  dimensions is treated.

**5.  $d$ -dimensional Gaussian sums: generalizations**

In this section, proofs or parts of proofs that are similar to those in two dimensions will not be repeated. Let us consider the case where non-diagonal elements  $\omega_{ij}$  ( $i \neq j$ ) are sufficiently small for the same vector  $\vec{\theta}$ , and where  $\Omega$  is invertible. We shall study the case  $\vec{a} = \vec{b} = \vec{0}$ . The general case can be deduced by analogy. Performing the dual transformation on only one variable  $k_m$  with  $1 \leq m \leq d$  on the sum defining

$$\begin{aligned}
 {}^d\sigma_{A_{\vec{a}}}^{[\vec{0}, \vec{0}]}(\Omega, \vec{\theta}) &= \sum_{k_1=0}^{q_1-1} \dots \sum_{k_d=0}^{q_d-1} \exp \left( i\pi \left[ \sum_{1 \leq i \leq d} \omega_{ii} k_i^2 + 2 \sum_{1 \leq i < j \leq d} \omega_{ij} k_i k_j + 2 \sum_{1 \leq i \leq d} k_i \theta_i \right] \right) \\
 &= \sum_{k_1=0}^{q_1-1} \dots \sum_{k_m=0}^{\omega_{m,m} q_m - 1} \dots \sum_{k_d=0}^{q_d-1} \exp \left( i\pi \left[ \sum_{1 \leq i \leq d}^{i \neq m} \omega_{ii} k_i^2 \right. \right. \\
 &\quad \left. \left. + 2 \sum_{1 \leq i < j \leq d}^{i \neq m, j \neq m} \omega_{ij} k_i k_j + 2 \sum_{1 \leq i \leq d}^{i \neq m} k_i \theta_i \right] \right) \\
 &\quad \times \sum_{k_m=0}^{q_m-1} \exp \left( i\pi \left[ \omega_{mm} k_m^2 + 2 \left( \theta_m + \sum_{1 \leq i \leq d} \omega_{mi} k_i \right) k_m \right] \right), \tag{5.1}
 \end{aligned}$$

we get

$$\begin{aligned}
 {}^d\sigma_{A_{\vec{a}}}^{[\vec{0}, \vec{0}]}(\Omega, \vec{\theta}) &= \sum_{k_1=0}^{q_1-1} \dots \sum_{k_m=0}^{q_m-1} \dots \sum_{k_d=0}^{q_d-1} \exp \left( i\pi \left[ \sum_{1 \leq i \leq d}^{i \neq m} \omega_{ii} k_i^2 + 2 \sum_{1 \leq i < j \leq d}^{i \neq m, j \neq m} \omega_{ij} k_i k_j \right] \right) \\
 &\quad \times \exp \left( 2 \sum_{1 \leq i \leq d}^{i \neq m} k_i \theta_i \right) \sqrt{\frac{i}{\omega_{mm}}} \sum_{k_i \neq k_m} \exp \left( -\frac{\pi}{\omega_{mm}} \left( \theta_m + \sum_{1 \leq i \leq d} \omega_{mi} k_i \right)^2 \right) \\
 &\quad \times \left( \sum_{k_m=0}^{\omega_{m,m} q_m - 1} \exp \left( i\pi \left[ -\frac{k_m^2}{\omega_{mm}} + 2 \left( \theta_m + \sum_{1 \leq i \leq d} \omega_{mi} k_i \right) \frac{k_m}{\omega_{mm}} \right] \right) \right) + \mathcal{O}(1). \tag{5.2}
 \end{aligned}$$

Let us define  $D_m$  as the dual transformation operator on the index  $k_m$ . We observe that for any function  $f$

$$\mathcal{O}_m(1) f(\dots, k_m, \dots) = \mathcal{O}(1) f(\dots, k_m = 0, \dots). \tag{5.3}$$

Thus regrouping terms using the previous remark, we have

$$\begin{aligned}
 {}^d\sigma_{A_{\vec{a}}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) &= \sqrt{\frac{i}{\omega_{mm}}} \exp \left( -i\pi \frac{\theta_m^2}{\omega_{mm}} \right) \sum_{k_1=0}^{q_1-1} \dots \left( \sum_{k_m=0}^{\omega_{m,m} q_m - 1} + \mathcal{O}_m(1) \right) \\
 &\quad \dots \sum_{k_d=0}^{q_d-1} \exp \left( -i\pi \left( \frac{k_m^2}{\omega_{mm}} - \sum_{i \neq m} \left( \omega_{ii} - \frac{\omega_{mi}^2}{\omega_{mm}} \right) k_i^2 \right) \right) \\
 &\quad \times \exp \left( i\pi \sum_{1 \leq i < j \leq d}^{i \neq m, j \neq m} \left( \omega_{ij} - \frac{\omega_{mi} \omega_{mj}}{\omega_{mm}} \right) k_i k_j \right) \exp \left( i\pi \sum_{1 \leq i \leq d}^{i \neq m} \frac{\omega_{mi}}{\omega_{mm}} k_i k_m \right)
 \end{aligned}$$



$$\times \exp \left( 2i\pi \sum_{1 \leq i \leq d}^{i \neq m} \left( \theta_1 - \frac{\omega_{mi}}{\omega_{mm}} \theta_m \right) k_i \right) \exp \left( 2i\pi \sum_{1 \leq i \leq d}^{i \neq m} \frac{\theta_m}{\omega_{mm}} k_m \right). \quad (5.4)$$

Now this relation may be re-expressed using the dualization with respect to the variables  $k_m, \mathcal{D}_m$ , defined by

$$\begin{aligned} \mathcal{D}_m & \left( \sum_{k_1=0}^{q_1-1} \dots \sum_{k_d=0}^{q_d-1} \exp \left( i\pi \left[ \sum_{1 \leq i \leq d} \omega_{ii} k_i^2 + 2 \sum_{1 \leq i < j \leq d} \omega_{ij} k_i k_j + 2 \sum_{1 \leq i \leq d} k_i \theta_i \right] \right) \right) \\ & = \Phi' \sum_{k_1=0}^{q_1-1} \dots \left( \sum_{k_m=0}^{\omega_{mm} q_m - 1} + \mathcal{O}_m(1) \right) \dots \sum_{k_d=0}^{q_d-1} \exp \left( i\pi \left[ \sum_{1 \leq i \leq d} \omega'_{ii} k_i^2 \right. \right. \\ & \quad \left. \left. + 2 \sum_{1 \leq i < j \leq d} \omega'_{ij} k_i k_j + 2 \sum_{1 \leq i \leq d} k_i \theta'_i \right] \right). \end{aligned} \quad (5.5)$$

Thus under the action of  $\mathcal{D}_m$ , matrix elements of  $\Omega, \vec{\theta}$  are transformed and a multiplicative factor is introduced, leading to the following relations:

- $\omega'_{ij} = \omega_{ij} - \frac{\omega_{im}\omega_{jm}}{\omega_{mm}}, \forall 1 \leq i \leq d, \forall 1 \leq j \leq d$  with  $i \neq m$  and  $j \neq m$
- $\omega'_{mm} = -\frac{1}{\omega_{mm}}$ ,
- $\theta'_i = \theta_i - \frac{\omega_{im}}{\omega_{mm}} \theta_m, \forall 1 \leq i \leq d$ , with  $i \neq m$
- $\theta'_m = \frac{\theta_m}{\omega_{mm}}$
- $\Phi' = \sqrt{\frac{i}{\omega_{mm}}} \exp(-i\pi \frac{\theta_m^2}{\omega_{mm}}) \Phi$  with  $\Phi = 1$

and of course we have  $\Omega' = \mathcal{D}_m(\Omega), \vec{\theta}' = \mathcal{D}_m(\vec{\theta})$  and  $\Phi' = \mathcal{D}_m(\Phi)$ . Under the action of duality on the index  $k_m$ , the matrix  $\Omega$  becomes a symmetric matrix. So let us apply this transformation successively as follows,  $1 \leq m \leq d$ ,

$$\begin{bmatrix} \Omega \\ \vec{\theta} \\ 1 \end{bmatrix} = \begin{bmatrix} \Omega_0 \\ \vec{\theta}_0 \\ \Phi_0 \end{bmatrix} \xrightarrow{\mathcal{D}_1} \begin{bmatrix} \Omega_1 \\ \vec{\theta}_1 \\ \Phi_1 \end{bmatrix} \xrightarrow{\mathcal{D}_2} \dots \xrightarrow{\mathcal{D}_m} \begin{bmatrix} \Omega_m \\ \vec{\theta}_m \\ \Phi_m \end{bmatrix} \quad (5.6)$$

where each bracket represents, with the set  $[\Omega, \vec{\theta}, \Phi]$  the effect of the transformation. Define now the symmetric matrix  $B_m$  taken from  $\Omega$  as

$$B_m = \begin{bmatrix} \omega_{11} & \dots & \omega_{1m} \\ \vdots & & \vdots \\ \omega_{m1} & \dots & \omega_{mm} \end{bmatrix} \quad (5.7)$$

$$\Phi_m = \frac{\sqrt{1^m}}{\sqrt{\det B_m}} e^{-i\pi \cdot (\theta_1, \dots, \theta_m) B_m^{-1} (\theta_1, \dots, \theta_m)} \quad (5.8)$$

with for  $1 \leq i \leq m$  and  $1 \leq j \leq m$

$$(\omega_{m+1})_{i,j} = \frac{(-1)^{i+j}}{\det B_m} \det \begin{bmatrix} \omega_{11} & \dots & \omega_{1,j-1} & \omega_{1,j+1} & \dots & \omega_{1,m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \omega_{i-1,1} & \dots & \omega_{i-1,j-1} & \omega_{i-1,j+1} & \dots & \omega_{i-1,m} \\ \omega_{i+1,1} & \dots & \omega_{i+1,j-1} & \omega_{i+1,j+1} & \dots & \omega_{i+1,m} \\ \vdots & & \vdots & \vdots & & \vdots \\ \omega_{m,1} & \dots & \omega_{m,j-1} & \omega_{m,j+1} & \dots & \omega_{m,m} \end{bmatrix} \quad (5.9)$$

that is essentially the  $(i, j)$  cofactor of  $B_m$ . In a more compact way we can write

$$\Omega_{m+1}|_{(i,j)=1,\dots,m} = -B_m^{-1} \tag{5.10}$$

for  $m < i$  and  $1 \leq j \leq m$

$$(\omega_{m+1})_{i,j} = \frac{1}{\det B_m} \det \begin{bmatrix} \omega_{11} & \dots & \omega_{1,j-1} & \omega_{1i} & \omega_{1,j+1} & \dots & \omega_{1,m} \\ \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \omega_{m,1} & \dots & \omega_{m,j-1} & \omega_{mi} & \omega_{m,j+1} & \dots & \omega_{m,m} \end{bmatrix}. \tag{5.11}$$

This is obtained by the substitution of the  $j$ th column by the  $i$ th line. Hence if  $m < i$  and  $m < j$ , we have

$$(\omega_{m+1})_{i,j} = \frac{1}{\det B_m} \det \begin{bmatrix} \omega_{11} & \dots & \omega_{1,m} & \omega_{i,j} \\ \vdots & & \vdots & \vdots \\ \omega_{m1} & \dots & \omega_{m,m} & \omega_{m,j} \\ \omega_{i1} & \dots & \omega_{i,m} & \omega_{i,j} \end{bmatrix}. \tag{5.12}$$

The right-hand side is an  $(m + 1) \times (m + 1)$  matrix obtained from  $B_m$  having a piece of the  $i$ th line and  $j$ th column with  $m < i$  and  $m < j$ . Finally

$$\begin{bmatrix} \theta_1^m \\ \vdots \\ \theta_m^m \end{bmatrix} = B_m^{-1} \begin{bmatrix} \theta_1 \\ \vdots \\ \theta_m \end{bmatrix} \tag{5.13}$$

and for all  $i > m$

$$\theta_i^m = \theta_i + \sum_{j=1}^m (\omega_{m+1})_{ij} \theta_j. \tag{5.14}$$

These results may be proven by recursion. The proof is uniquely based on the expansion of the determinant into minors. For example for  $(\omega_{m+1})_{11}$ , one has to compute

$$(\omega_m)_{11} - \frac{(\omega_m)_{1m}^2}{(\omega_m)_{mm}} \tag{5.15}$$

with

$$(\omega_m)_{11} = \frac{(\tilde{B}_m)_{11}}{\det B_m} \tag{5.16}$$

where  $\tilde{B}_m$  is the matrix formed by cofactors of  $B_m$

$$(\omega_m)_{mm} = \frac{\det B_{m+1}}{\det B_m} \tag{5.17}$$

with  $\det B_{m+1} = \omega_{m+1,m+1} \det B_m + \sum_{i=1}^m \sum_{j=1}^m \omega_{i,m+1} \omega_{m+1,j} (\tilde{B}_m)_{i,j}$

$$(\omega_m)_{im} = \sum_{i=1}^m \omega_{i,m+1} \frac{(\tilde{B}_m)_{i,1}}{\det B_m}. \tag{5.18}$$

Hence

$$\begin{aligned} (\omega_m)_{11} - \frac{(\omega_m)_{1m}^2}{(\omega_m)_{mm}} &= \frac{\omega_{m+1,m+1} (\tilde{B}_m)_{11}}{\det B_{m+1}} \\ &+ \frac{1}{\det B_{m+1} \det B_m} \sum_{i=1}^m \sum_{j=1}^m \omega_{i,m+1} \omega_{m+1,j} ((\tilde{B}_m)_{11} (\tilde{B}_m)_{ij} - (\tilde{B}_m)_{1j} (\tilde{B}_m)_{i1}). \end{aligned} \tag{5.19}$$

Knowing that the determinant is invariant (up to a sign) under permutations of lines or columns, it is sufficient to compute

$$\begin{aligned}
 (\tilde{B}_m)_{11}(\tilde{B}_m)_{mm} - (\tilde{B}_m)_{1m}(\tilde{B}_m)_{m1} &= \left( \sum_{\sigma \in \mathcal{S}_{m-1}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{m-1} \omega_{i+1, \sigma(i)+1} \right) \\
 &\times \left( \sum_{\sigma' \in \mathcal{S}_{m-1}} (-1)^{\epsilon(\sigma')} \prod_{i=1}^{m-1} \omega_{i, \sigma'(i)} \right) - \left( \sum_{\sigma \in \mathcal{S}_{m-1}} (-1)^{\epsilon(\sigma')} \prod_{i=1}^{m-1} \omega_{i, \sigma(i)+1} \right) \\
 &\times \left( \sum_{\sigma' \in \mathcal{S}_{m-1}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{m-1} \omega_{i+1, \sigma'(i)} \right). \tag{5.20}
 \end{aligned}$$

Where  $\mathcal{S}_{m-1}$  is the symmetric group of permutations of order  $(m - 1)$ . After expansion and regrouping, we get the double sum

$$\sum_{\sigma \in \mathcal{S}_{m-1}} \sum_{\sigma' \in \mathcal{S}_{m-1}} (-1)^{\epsilon(\sigma)+\epsilon(\sigma')} \prod_{i=1}^{m-1} \prod_{j=1}^{m-1} (\omega_{i+1, \sigma(i)+1} \omega_{j, \sigma'(j)} - \omega_{i, \sigma(i)+1} \omega_{j+1, \sigma'(j)}). \tag{5.21}$$

Now the product of permutations  $\sigma \otimes \sigma' \in \mathcal{S}_{m-1} \times \mathcal{S}_{m-1}$  may be represented as  $\sigma \otimes \sigma' \in \mathcal{S}_m \times \mathcal{S}_{m-2}$  and the previous double sum is recast under the form

$$\begin{aligned}
 &\sum_{\sigma \in \mathcal{S}_m} \sum_{\sigma' \in \mathcal{S}_{m-2}} (-1)^{\epsilon(\sigma)+\epsilon(\sigma')} \prod_{i=1}^m \prod_{j=1}^{m-2} (\omega_{i, \sigma(i)} \omega_{j+1, \sigma'(j)+1}) \\
 &= \left( \sum_{\sigma \in \mathcal{S}_m} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{m\omega_{i, \sigma(i)}} \right) \left( \sum_{\sigma \in \mathcal{S}_{m-2}} (-1)^{\epsilon(\sigma)} \prod_{i=1}^{m-2} \omega_{i+1, \sigma(i)+1} \right) \\
 &= \det B_m \det \begin{bmatrix} \omega_{12} & \dots & \omega_{1, m-1} \\ \vdots & & \vdots \\ \omega_{m-1, 2} & \dots & \omega_{m-1, m-1} \end{bmatrix}. \tag{5.22}
 \end{aligned}$$

For opposite permutations of lines and columns we have the following relation:

$$\begin{aligned}
 &(\tilde{B}_m)_{ij}(\tilde{B}_m)_{kk'} - (\tilde{B}_m)_{ik}(\tilde{B}_m)_{jk'} \\
 &= \det B_m \det \begin{bmatrix} \omega_{11} & \dots & \hat{\omega}_{1, j} & \dots & \hat{\omega}_{1, k'} & \dots & \omega_{1, m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \hat{\omega}_{i1} & \dots & \hat{\omega}_{i, j} & \dots & \hat{\omega}_{i, k'} & \dots & \hat{\omega}_{i, m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \hat{\omega}_{k1} & \dots & \hat{\omega}_{k, j} & \dots & \hat{\omega}_{k, k'} & \dots & \hat{\omega}_{k, m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \omega_{m1} & \dots & \hat{\omega}_{m, j} & \dots & \hat{\omega}_{m, k'} & \dots & \omega_{m, m} \end{bmatrix}. \tag{5.23}
 \end{aligned}$$

The last matrix is of size  $(m - 2) \times (m - 2)$  obtained from  $B_m$  by the removal of columns  $j$  and  $k'$  and of the lines  $i$  and  $k$ . Consequently

$$(\omega_m)_{11} - \frac{(\omega_m)_{1m}^2}{(\omega_m)_{mm}} = \frac{(\tilde{B}_m)_{11}\omega_{m+1,m+1}}{\det B_{m+1}} + \left( \sum_{i=1}^m \sum_{j=1}^m \frac{\omega_{i,m+1}\omega_{j,m+1}}{\det B_{m+1}} \det \begin{bmatrix} \omega_{11} & \dots & \hat{\omega}_{1,j} & \dots & \hat{\omega}_{1,k'} & \dots & \omega_{1,m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \hat{\omega}_{i1} & \dots & \hat{\omega}_{i,j} & \dots & \hat{\omega}_{i,k'} & \dots & \hat{\omega}_{i,m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \hat{\omega}_{k1} & \dots & \hat{\omega}_{k,j} & \dots & \hat{\omega}_{k,k'} & \dots & \hat{\omega}_{k,m} \\ \vdots & & \vdots & & \vdots & & \vdots \\ \omega_{m1} & \dots & \hat{\omega}_{m,j} & \dots & \hat{\omega}_{m,k'} & \dots & \omega_{m,m} \end{bmatrix} \right). \tag{5.24}$$

But this expression corresponds to the expansion of the determinant of

$$(\omega_m)_{11} = \frac{(\tilde{B}_{m+1})_{11}}{\det B_{m+1}} = \det \begin{bmatrix} \omega_{22} & \dots & \omega_{2,m+1} \\ \vdots & & \vdots \\ \omega_{m+1,2} & \dots & \omega_{m+1,m+1} \end{bmatrix} \tag{5.25}$$

with respect to line  $(m + 1)$  and column  $(m + 1)$ . Other results may be proved along the same line, i.e. using the expansion of the determinant into minors. Hence if  $\Omega$  is invertible, one can perform the duality transformation with respect to all indices  $k_j$ , and the constants have the expressions

$$\Phi_d = \frac{\sqrt{\Gamma^d}}{\sqrt{\det \Omega}} e^{-i^t \vec{\theta} \Omega^{-1} \vec{\theta}} \quad \Omega_d = -\Omega^{-1} \quad \vec{\theta}_d = \Omega^{-1} \vec{\theta}. \tag{5.26}$$

Each sum  $\sum_{k_i}$  is replaced by  $\sum_{k_i} + \mathcal{O}_i(1)$ , as seen before. Proceeding as for  $d = 2$ , one can extend the two-dimensional result to the set of invertible matrices with non-integer diagonal elements. Due to linearity, this formula remains valid for sums on a bounded domain  $\mathcal{D} \in \mathbb{Z}^d$ . The hypothesis of a bounded domain is necessary since otherwise the Gaussian sums are not converging. This is not the case for theta functions. Consequently, the result is also valid for matrices  $\Omega \in M_d(]0, 1[)_{\text{inv}}$ .

There remains the extension to  $\mathbb{R} \setminus \mathbb{Z}$  to be examined. For this, let us introduce  $L_{ij} \in M_d(\{0, 1\})$  such that  $(L_{ij})_{k,k'} = 0$  for all  $k, k'$  except for  $(k = i, k' = j)$  or  $(k = j, k' = i)$ . Compute now

$$\begin{aligned}
 {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega + L_{ij}, \vec{\theta}) &= \sum_{\vec{k} \in \mathcal{D}} \exp \left( i\pi \left( {}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) (\Omega + L_{ij}) \left( \vec{n} - \frac{\vec{a}}{2} \right) + 2^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \left( \vec{\theta} - \frac{\vec{b}}{2} \right) \right) \right) \\
 &= \sum_{\vec{k} \in \mathcal{D}} \exp \left( i\pi \left( {}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \Omega \left( \vec{n} - \frac{\vec{a}}{2} \right) + 2^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \left( \vec{\theta} - \frac{\vec{b}}{2} \right) \right. \right. \\
 &\quad \left. \left. + {}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) L_{ij} \left( \vec{n} - \frac{\vec{a}}{2} \right) \right) \right)
 \end{aligned}$$

$$\begin{aligned}
 &= i^{a_i a_j} \sum_{\vec{k} \in \mathcal{D}} \exp\left(i\pi \left( {}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \Omega \left( \vec{n} - \frac{\vec{a}}{2} \right) \right. \right. \\
 &\quad \left. \left. + 2^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \left( \vec{\theta} - \frac{\vec{b}}{2} \right) - (a_i n_j + a_j n_i) \right) \right) \tag{5.27}
 \end{aligned}$$

since

$${}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) L_{ij} \left( \vec{n} - \frac{\vec{a}}{2} \right) = 2 \left( n_i - \frac{a_i}{2} \right) \left( n_j - \frac{a_j}{2} \right) = 2n_i n_j - (a_i n_j + a_j n_i) + \frac{a_i a_j}{2}. \tag{5.28}$$

We set  $\vec{b}' \equiv \vec{b} + a_i \vec{e}_j + a_j \vec{e}_i \pmod{(1, 1)}$  and obtain

$$\begin{aligned}
 {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega + L_{ij}, \vec{\theta}) &= (-i)^{a_i a_j} \sum_{\vec{k} \in \mathcal{D}} \exp\left(i\pi \left( {}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \Omega \left( \vec{n} - \frac{\vec{a}}{2} \right) \right. \right. \\
 &\quad \left. \left. + 2^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \left( \vec{\theta} - \frac{\vec{b}'}{2} \right) \right) \right) \\
 &= (-i)^{a_i a_j} {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}']}(\Omega, \vec{\theta}). \tag{5.29}
 \end{aligned}$$

Now let  $L_i \in M_d(\{0, 1\})_{\text{sym}}$  and  $(L_i)_{k,k'} = 0$  except for  $k = i, k' = 1$ . Then

$$\begin{aligned}
 {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega + L_i, \vec{\theta}) &= (-1)^{(1-a_i)n_i} (\sqrt{i})^{a_i} \sum_{\vec{k} \in \mathcal{D}} \exp\left(i\pi \left( {}^t \left( \vec{n} - \frac{\vec{a}}{2} \right) (\Omega) \left( \vec{n} - \frac{\vec{a}}{2} \right) \right. \right. \\
 &\quad \left. \left. + 2^t \left( \vec{n} - \frac{\vec{a}}{2} \right) \left( \vec{\theta} - \frac{\vec{b}}{2} \right) \right) \right) \tag{5.30}
 \end{aligned}$$

but with  $\vec{b}' = \vec{b} + (1 - a_i)\vec{e}_j$ , we have the result

$${}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega + L_i, \vec{\theta}) = (\sqrt{i})^{a_i} {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}']}(\Omega, \vec{\theta}), \tag{5.31}$$

and may state the theorem:

**Theorem 5.1.** *With the convention  $\sqrt{-1} = i$  and  $\forall \Omega \in M_d^*([0, 1]_{\text{inv}})$  and  $\forall \vec{\theta} \in ]0, 1[^d$ , the Gaussian sum in  $d$  dimensions admits the duality formula:*

•

$${}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \frac{(\sqrt{i})^d}{\det \Omega} e^{-i\pi {}^t \vec{\theta} \Omega^{-1} \vec{\theta}} \sum_{0 \leq i_1, \dots, i_d \leq d} \mathcal{O}(1)^{i_1 + \dots + i_d} {}^d \sigma_{\mathcal{P}_1^{i_1} \dots \mathcal{P}_d^{i_d} \Omega \mathcal{D}}^{[\vec{a}, \vec{b}]}(-\Omega^{-1}, \Omega^{-1} \vec{\theta}). \tag{5.32}$$

We also use the convention  $\mathcal{P}^0 = \mathbb{I}d$ , the identity map.

• For  $L_{ij} \in M_d(\{0, 1\})_{\text{sym}}$  introduced before and  $\vec{b}' \equiv \vec{b} + a_i \vec{e}_j + a_j \vec{e}_i \pmod{(1, 1)}$  we also have

$${}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega + L_{ij}, \vec{\theta}) = (-i)^{a_i a_j} {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}']}(\Omega, \vec{\theta}). \tag{5.33}$$

• And for  $L_i \in M_d(\{0, 1\})_{\text{sym}}$  introduced before and  $\vec{b}' = \vec{b} + (1 - a_i)\vec{e}_j$ , we obtain

$${}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega + L_i, \vec{\theta}) = (\sqrt{i})^{a_i} {}^d \sigma_{\mathcal{D}}^{[\vec{a}, \vec{b}']}(\Omega, \vec{\theta}). \tag{5.34}$$

The functions  ${}^d C_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta})$  and  ${}^d S_{\mathcal{D}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta})$  verify similar duality relations.

If  $\Omega$  is not invertible, one may perform a partial duality transformation. Up to a permutation of indices, the duality transformation may be applied to those of the indices for which the minors are nonzero and the number of indices corresponds simply to the rank of the matrix  $\Omega$ . For the remaining indices the summation can be performed, since they are geometrical sums.

**6. Applications to the computation of  $\xi_n[\beta, x; P_i]$**

Now we go back to equation (3.7) which we shall write using the notation of the previous section. The matrix  $\Omega$  belongs to  $M_{2q-1}^*(\mathbb{R})_{\text{sym}}$  and has the form

$$\Omega = \frac{\beta}{\sum P_i} \begin{bmatrix} P_2 (\sum P_i - P_2) & -P_2 P_3 & \dots & -P_2 P_{2q} \\ -P_2 P_3 & P_3 (\sum P_i - P_3) & \vdots & -P_3 P_{2q} \\ \vdots & \vdots & \ddots & \vdots \\ -P_2 P_{2q} & -P_3 P_{2q} & \dots & P_{2q} (\sum P_i - P_{2q}) \end{bmatrix} \tag{6.1}$$

we note  $\sum$  for  $\sum_{i=2}^{2q}$  and

$$\vec{\theta} = x \begin{bmatrix} P_2 \\ P_3 \\ \vdots \\ P_{2q} \end{bmatrix} \in \text{Vect}_{2q-1}(\mathbb{R}). \tag{6.2}$$

It is easy to see that  $\det \Omega = 0$  ( $\text{rank}(\Omega) \leq 2(q - 1)$ ). By just adding all the lines, we get a line of 0. Let us compute the minor

$$\begin{aligned} \det \frac{\beta}{\sum P_i} & \begin{bmatrix} P_2 (\sum P_i - P_2) & -P_2 P_3 & \dots & -P_2 P_{2q-1} \\ -P_2 P_3 & P_3 (\sum P_i - P_3) & \vdots & -P_3 P_{2q-1} \\ \vdots & \vdots & \ddots & \vdots \\ -P_2 P_{2q-1} & -P_3 P_{2q-1} & \dots & P_{2q-1} (\sum P_i - P_{2q-1}) \end{bmatrix} \\ &= \beta^{2(q-1)} \frac{P_2 \dots P_{2q-1}}{(\sum P_i)^{2(q-1)}} \det \begin{bmatrix} \sum P_i - P_2 & -P_2 & \dots & -P_2 \\ -P_3 & \sum P_i - P_3 & \vdots & -P_3 \\ \vdots & \vdots & \ddots & \vdots \\ -P_{2q-1} & -P_{2q-1} & \dots & \sum P_i - P_{2q-1} \end{bmatrix}. \end{aligned} \tag{6.3}$$

Now replacing the first line by the sum of all the lines, we obtain

$$= \frac{\beta^{2(q-1)} P_2 \dots P_{2q}}{(\sum P_i)^{2(q-1)}} \det \begin{bmatrix} 1 & 1 & \dots & 1 \\ -P_3 & \sum P_i - P_3 & \vdots & -P_3 \\ \vdots & \vdots & \ddots & \vdots \\ -P_{2q-1} & -P_{2q-1} & \dots & \sum P_i - P_{2q-1} \end{bmatrix}. \tag{6.4}$$

Replacing the  $j$ th column by the difference of the  $j$ th and the first column. Doing it for all  $j \geq 2$ , we get to the expression

$$\begin{aligned}
 &= \beta^{2(q-1)} \frac{P_2 \cdots P_{2q}}{(\sum P_i)^{2(q-1)}} \det \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ -P_3 & \sum P_i & 0 & \vdots & 0 \\ -P_4 & 0 & \sum P_i & \vdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -P_{2q-1} & 0 & 0 & \cdots & \sum P_i \end{bmatrix} \\
 &= \beta^{2(q-1)} \frac{P_2 \cdots P_{2q}}{(\sum P_i)} \neq 0.
 \end{aligned} \tag{6.5}$$

Consequently we have the lemma

**Lemma 10.**

$$\text{rank}(\Omega) = 2(q - 1). \tag{6.6}$$

This result implies the duality transformation of  $\xi_n$  only with respect to indices  $k_2, \dots, k_{2q-1}$ . Thus we have

$$\Phi_{2(q-1)} = \left(\frac{i}{\beta}\right)^{(q-1)} \sqrt{\frac{\sum P_i}{P_2 \cdots P_{2q}}} \exp(-i\pi^t(\theta_2, \dots, \theta_{2q-1})\Omega_{2(q-1)}^{-1}(\theta_2, \dots, \theta_{2q-1})) \tag{6.7}$$

with

$$\Omega_{|i,j=2,\dots,2q-1} \stackrel{\text{def}}{=} \bar{\Omega} = \frac{1}{\sum P_i} \begin{bmatrix} P_2(\sum P_i - P_2) & \cdots & -P_2 P_{2q-1} \\ \vdots & \ddots & \vdots \\ -P_2 P_{2q-1} & \cdots & P_{2q-1}(\sum P_i - P_{2q-1}) \end{bmatrix}. \tag{6.8}$$

Using the results of the above section we also obtain  $\forall i = 2, \dots, 2q - 1$

$$\begin{aligned}
 \omega_{2q-1|i,2q} &= \frac{1}{(\sum P_i)^{2q-3} P_2 \cdots P_{2q}} \\
 &\times \det \begin{bmatrix} P_2(\sum P_i - P_2) & \cdots & -P_2 P_{2q} & \cdots & -P_2 P_{2q-1} \\ -P_2 P_3 & \cdots & -P_3 P_{2q} & \cdots & -P_2 P_{2q-1} \\ \vdots & \ddots & \vdots & & \vdots \\ -P_2 P_i & \cdots & -P_{2q} P_i & \cdots & -P_i P_{2q-1} \\ \vdots & & \vdots & \ddots & \vdots \\ -P_2 P_{2q-1} & \cdots & -P_{2q-1} P_{2q} & \cdots & -P_{2q-1}(\sum P_i - P_{2q-1}) \end{bmatrix}.
 \end{aligned} \tag{6.9}$$

In the last matrix we substitute the  $i$ th column by the sum of all the columns of the matrix such that the  $i$ th column becomes

$$[-P_i P_2, \dots, P_i(P_1 + \dots + \hat{P}_i + \dots P_{2q}), \dots, -P_i P_{2q-1}]. \tag{6.10}$$

So that  $\omega_{2q-1|i,2q} = 1$ . We also have

$$\begin{bmatrix} \theta_2^{2q-1} \\ \vdots \\ \theta_{2q-1}^{2q-1} \end{bmatrix} = \bar{\Omega}^{-1} \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_{2q-1} \end{bmatrix} \quad \theta_{2q-1}^{2q} = \theta_{2q} + \dots + \theta_2 \tag{6.11}$$

At the end we get the form of  $\Omega_{2q-2}$  as

$$\Omega_{2q-2} = \begin{bmatrix} & & & 1 \\ & (\bar{\Omega}^{-1}) & & \vdots \\ & & & 1 \\ 1 & \dots & 1 & 0 \end{bmatrix} \tag{6.12}$$

where  $\bar{\Omega}$  is the  $(2q - 2) \times (2q - 2)$  matrix defined above. So considering all the results modulo 1, the matrix  $\Omega_{2q-2}$  does not admit coupling between  $k_{2q}$  and  $k_i$  for  $i = 2, \dots, 2q - 1$ . Consequently with

$$A_{\bar{n}} = \underbrace{A_{n-1, \dots, n-1}}_{2q-2 \text{ times}}$$

and  $\theta_i = x P_i, \forall i = 2, \dots, 2q - 1$

$$\begin{aligned} \mathcal{D}_{2q-1} \circ \dots \circ \mathcal{D}_2(\xi_n(\beta, x; P_2, \dots, P_{2q})) &= \left(\frac{i}{\beta}\right)^{q-1} \sqrt{\frac{\sum_{i=1}^{2q} P_i}{P_2 \dots P_{2q}}} \\ &\times e^{-i\pi'(\theta_2, \dots, \theta_{2q-1})\bar{\Omega}^{-1}(\theta_2, \dots, \theta_{2q-1})} \sum_{k_1=0}^{n-1} \sum_{k_{2q}=0}^{n-1} e^{2\pi x(k_{2q}-k_1)\sum P_i} \\ &\times \sum_{2 \leq i_2, \dots, i_{2q-1} \leq 2q-1} \mathcal{O}(1)^{i_2+\dots+i_{2q-1}} \cdot 2^{q-2} \sigma_{P_{i_2} \circ \dots \circ P_{i_{2q-1}}[\Omega_{A_{\bar{n}}}]^{[0, \vec{0}]}} \left(-\frac{\bar{\Omega}^{-1}}{\beta}, \frac{\bar{\Omega}^{-1}\vec{\theta}}{\beta}\right) \end{aligned} \tag{6.13}$$

where  $\vec{\theta}$  stands for the vector

$$\vec{\theta} = \begin{bmatrix} \theta_2 \\ \vdots \\ \theta_{2q-1} \end{bmatrix}. \tag{6.14}$$

Hence

$$\begin{aligned} \xi_n(\beta, x; P_2, \dots, P_{2q}) &= \mathcal{O}(1) \sqrt{\frac{\sum_{i=1}^{2q} P_i}{P_2 \dots P_{2q}}} \sum_{k_1}^{n-1} \sum_{k_{2q}}^{n-1} e^{2\pi x(k_{2q}-k_1)\sum P_i} \\ &\times \sum_{2 \leq i_2, \dots, i_{2q-1} \leq 2q-1} \sigma_{P_{i_2} \circ \dots \circ P_{i_{2q-1}}[\Omega_{A_{\bar{n}}}]^{[0, \vec{0}]}} (-\bar{\Omega}^{-1}, \bar{\Omega}^{-1}\vec{\theta}). \end{aligned} \tag{6.15}$$

Here we are reduced to computing the  $\sigma$ -functions, with  $\Omega$  a matrix with integer entries of the form

$$\sigma_{A_{\bar{n}}}^{[0, \vec{0}]}(\lambda\Omega, \vec{\theta}) = \sum_{\vec{k} \in A_{\bar{n}}} e^{i\pi(\lambda' \vec{k} \Omega \vec{k} + \vec{k} \vec{\theta})} \tag{6.16}$$

where  $\lambda \in \mathbb{R}$ .

**Lemma 11.** For any matrix  $\Omega$  with integer entries, there exists a basis  $(\vec{k}')$  such that the quadratic form  ${}^t \vec{k} \Omega \vec{k}$  becomes

$${}^t \vec{k} \Omega \vec{k} = \lambda_1 k_1'^2 + \dots + \lambda_{2q-2} k_{2q-2}'^2 = {}^t \vec{k}' \Lambda \vec{k}' \tag{6.17}$$

where  $\Lambda$  is a diagonal matrix with rational entries. If we note  $\vec{k}' = U\vec{k}$ ,  $U$  has also integer entries.



**Proof.** Let us write down explicitly the expression of the quadratic form

$$\begin{aligned}
 \sum_{ij} \omega_{i,j} k_i k_j &= \sum_{1 \leq i \leq d} \omega_{ii} k_i^2 + 2 \sum_{1 \leq i < j \leq d} \omega_{ij} k_i k_j \\
 &= \omega_{mm} \left( k_m + \sum_{i=1, i \neq m}^d \frac{\omega_{mi}}{\omega_{mm}} k_i \right)^2 - \omega_{mm} \left( \sum_{i=1, i \neq m}^d \frac{\omega_{mi}}{\omega_{mm}} k_i \right)^2 \\
 &\quad + 2 \sum_{1 \leq i < j \leq d, (i,j) \neq m} \omega_{ij} k_i k_j \\
 &= \omega_m k_m'^2 + \sum_{i=1, i \neq m} \omega'_{ii} k_i'^2 + 2 \sum_{1 \leq i < j \leq d, (i,j) \neq m} \omega'_{ij} k_i' k_j'. \tag{6.18}
 \end{aligned}$$

So by redefining the  $\omega'_{ij}$  and proceeding by induction we obtained the expected result.  $\square$

The transition matrix  $U$  used for the change of basis can be taken with integer entries instead of rational entries. Thus  $\forall i = 1, \dots, d, k'_i = \sum_{j=1}^d B_{ij} k_j$  and  $U$  is of course invertible. Moreover, up to a permutation of indices one can ensure that the next minor is non-zero. We then set

$$U' = \begin{bmatrix} U_{11} & \dots & U_{1,d-1} \\ \vdots & & \vdots \\ U_{d-1,1} & \dots & U_{d-1,d-1} \end{bmatrix} \tag{6.19}$$

and  $\det(U') \neq 0$ . Let  $\mathcal{D}$  be a bounded domain of  $\mathbb{N}^d$  and let  $\vec{P} \in U(\mathcal{D})$ . Then  $\vec{P}$ , by construction, is a vector with integer components and  $\exists \vec{k} \in \mathcal{D}$  such that  $\vec{P} = U\vec{k}$ . At this stage, one should show that  $U(\mathcal{D})$  can be decomposed into integer sub-lattices  $R_i$  of the lattice spacing  $\delta_i \in \mathbb{N}$ . Thus  $\vec{P} \in R_i$ , then  $\vec{P} + \delta_i \vec{e}_j \in R_i$  with  $\vec{e}_j, j = 1, \dots, d$ , being a unit vector.  $R_i$  is defined inside the boundary of  $U(\mathcal{D})$ . To prove this last property one must solve in  $\mathbb{N}^d$ , the system  $\forall i = 1, \dots, d - 1$

$$\sum_{j=1}^d B_{ij} k_j = 0 \quad \sum_{j=1}^d B_{dj} k_j = \delta. \tag{6.20}$$

We have thus

$$U' \begin{bmatrix} k_1 \\ \vdots \\ k_{d-1} \end{bmatrix} = -k_d \begin{bmatrix} U_{1,d} \\ \vdots \\ U_{1,d-1} \end{bmatrix} \tag{6.21}$$

and as  $\det U' \neq 0$  then

$$\begin{bmatrix} k_1 \\ \vdots \\ k_{d-1} \end{bmatrix} = -\frac{k_d}{\det U'} \tilde{U}' \begin{bmatrix} U_{1,d} \\ \vdots \\ U_{1,d-1} \end{bmatrix} \tag{6.22}$$

where  $\tilde{U}'$  is a matrix of the cofactor of  $U'$  (with integer entries). Let us take  $k_d = -\det U'$  so we have determined in  $\mathbb{N}^d$  the solution the above equations

$$\begin{bmatrix} k_1 \\ \vdots \\ k_{d-1} \end{bmatrix} = \tilde{U}' \begin{bmatrix} U_{1,d} \\ \vdots \\ U_{1,d-1} \end{bmatrix}. \tag{6.23}$$

It is clear that any multiple of this vector is a solution. Let us compute now  $\delta$ , defined by

$$\delta = \sum_{j=1}^d U_{dj}k_j = U_{dd} \det U' + \sum_{i=1}^{d-1} \sum_{j=1}^{d-1} U_{dj} \tilde{U}'_{ij} U_{id}. \tag{6.24}$$

Thus  $\delta = \det U$  by expanding with respect to the last line and column, we obtain

**Proposition 12.** *Let  $U(\mathcal{D})$  be a bounded domain of  $\mathbb{N}^d$  and  $U$  an integer invertible matrix. Then there exists at most  $\delta^d$  hypercubes  $R_i \in \mathbb{N}^d$  of the lattice spacing  $\delta_i$  ( $\delta_i$  being a divisor of  $\delta$ , with  $\delta = \det U$ ), such that  $\forall P \in U(\mathcal{D}), \exists 1 \leq i \leq \delta^d$  for which  $P \in R_i$ . The hypercubes  $(R_i)_{i=1, \dots, \delta^d}$  are all limited by boundaries of  $U(\mathcal{D})$ .*

**Proposition 13.** *Let  $(\mathcal{F}_n)_{n \in \mathbb{N}}$  be a family of hypercubes of  $\mathbb{N}^d$ , defined by  $\vec{k} = (k_1, \dots, k_d) \in \mathcal{F}_n$  with  $\forall 1 \leq i \leq d, k_i \in \mathbb{N}$  and  $1 \leq k_i \leq n$ . Let  $(\mathcal{D}_n)_{n \in \mathbb{N}}$  be an increasing sequence of connected domains of  $\mathbb{N}^d$  such that there exists  $t \in \mathbb{R}$  for which  $\forall n \in \mathbb{N}: \mathcal{F}_n \subset \mathcal{D}_n \subset \mathcal{F}_{tn}$ . Then, uniformly with respect to  $\theta$*

$${}^d\sigma_{\mathcal{D}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) \underset{n \rightarrow \infty}{\sim} {}^d\sigma_{\mathcal{F}_n}^{[\vec{0}, \vec{0}]}(\Omega, \vec{\theta}). \tag{6.25}$$

**Proof.**  $\mathbb{N}^d$  as a partially ordered set, we can easily prove that  $\forall (\vec{a}, \vec{b}) \in \text{Vect}_d(\{0, 1\}), \forall \Omega \in M_d(\mathbb{R})$  and  $\forall \vec{\theta} \in \text{Vect}_d(\mathbb{R})$  the mapping  ${}^d\sigma_{(\cdot)}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta})$  defined as follows is increasing:

$$\begin{aligned} & {}^d\sigma_{(\cdot)}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) : \mathcal{P}(\mathbb{N}^d) \longrightarrow \mathbb{C} \\ & \mathcal{F} \longmapsto {}^d\sigma_{\mathcal{F}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}). \end{aligned} \tag{6.26}$$

So for  $n$  sufficiently large we have  ${}^d\sigma_{\mathcal{F}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) \leq {}^d\sigma_{\mathcal{D}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) \leq {}^d\sigma_{\mathcal{F}_{tn}}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta})$ . Now taking into account that  ${}^d\sigma_{\mathcal{F}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \mathcal{O}({}^d\sigma_{\mathcal{F}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}))$ , we can conclude that  ${}^d\sigma_{\mathcal{D}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}) = \mathcal{O}({}^d\sigma_{\mathcal{F}_n}^{[\vec{a}, \vec{b}]}(\Omega, \vec{\theta}))$  and is homogeneous in  $\vec{\theta}$ .  $\square$

**Remark 14.** Here we just recall the result of [DM] that  $\forall \chi \in [1/2, 1] \exists x \in \mathbb{R}$  such that  ${}^1\sigma_n^{[0,0]}(x, 0) = \mathcal{O}(n^x)$ .

According to [HL, p 202], the hypothesis of uniformity is automatically verified in Gaussian sums with  $d = 1$ . This hypothesis remains valid for  $d$ -dimensional Gaussian sums. Let  $R_1(\vec{n}), \dots, R_d(\vec{n})$  be the sub-lattices making up  $B(\vec{\Omega}A_{\vec{n}})$ . Then we have

$$\begin{aligned} & \sum_{0 \leq i_2, \dots, i_{2q-2} \leq 2q-1} \mathcal{O}(1)^{i_2 + \dots + i_{2q-1}} \cdot {}^{2q-2}\sigma_{\mathcal{P}_{i_2 \circ \dots \circ \mathcal{P}_{i_{2q-1}}}(\vec{\Omega}A_{\vec{n}})}^{[\vec{0}, \vec{0}]} \left( -\frac{\vec{\Omega}^{-1}}{\beta}, \frac{\vec{\Omega}^{-1}\vec{\theta}}{\beta} \right) \\ & = \sum_{j=1}^s \sum_{\substack{i_2 \in [1, 2q-1], \dots, \\ i_{2q-2} \in [1, 2q-1]}} \mathcal{O}(1)^{i_2 + \dots + i_{2q-1}} \cdot {}^{2q-2}\sigma_{\mathcal{P}_{i_2 \circ \dots \circ \mathcal{P}_{i_{2q-1}}}(R_j)}^{[\vec{0}, \vec{0}]} \left( -\frac{\Lambda^{-1}}{\beta}, -\frac{\Lambda^{-1}\vec{\theta}'}{\beta} \right) \end{aligned} \tag{6.27}$$

with  $\vec{\theta}' = B^{-1}\vec{\theta}$ , where  $B$  is the matrix of the basis change described in lemma 11 such that  $\Lambda$  is a diagonal matrix with integer (or rational) entries. Since  $\Lambda$  has integer matrix elements, it follows from the above formula that there exists a finite number of sub-lattices for which the multi-variable sum is reduced to a product of  $(2q - 2)$  factors of one-dimensional Gaussian sums where each factor is of the form  ${}^1\sigma_{(R_j)}^{[\vec{0}, \vec{0}]}(-\frac{1}{\lambda\beta}, -\frac{1}{\lambda\beta}\theta)$  where  $\lambda$  is an eigenvalue of  $\Lambda$ . The following section will consider the problem of the asymptotic behaviour of such Gaussian sums.

### 6.1. Estimations of Gaussian sums using the duality formula

We shall now present the theorem which motivates the previous construction. Let us recall the method of Hardy and Littelwood using the duality formula for the computation of the Gaussian sums. They give the expression,  $\forall x \in ]0, 1[, \forall \theta \in ]0, 1[$  and  $\forall (a, b) \in \{0, 1\}$ , and any  $n \in \mathbb{N}$ :

$${}^1\sigma_n^{[a,b]}(x, \theta) = \mathcal{O}(1)\sqrt{nx_1 \cdots x_\nu} + \frac{\mathcal{O}(1)}{\sqrt{xx_1 \cdots x_\nu}}. \quad (6.28)$$

Recall briefly the steps: for convenience let us denote  $x_0 = x$ .  $\forall i = 0, \dots, \nu$ , we have  $0 < x_i < 1$ .  $x_i$  are the rest of the development in continuous fractions of  $x$ :

$$x = \frac{1}{a_1 + \frac{1}{a_2 + \cdots}} \stackrel{\text{not.}}{=} [a_1, a_2, \dots, ] \quad (6.29)$$

and  $\forall i > 0$  the  $(x_i)_{i \in \mathbb{N}}$  are given inductively by the relation

$$x_{i-1} = \frac{1}{a_i + x_i}. \quad (6.30)$$

Hence, for any  $n$ , there is an integer  $\nu$  determined in such a way that

$$nx_0 \cdots x_\nu \leq 1 \leq nx_0 \cdots x_{\nu-1}. \quad (6.31)$$

Through this condition, the first term on the rhs in equation (6.28) becomes irrelevant with respect to the second one. Let us put  $\nu = \nu(n)$  in the above expression. So there exists a positive constant  $H$  such that

$$x_0 \cdots x_\nu \geq H a_{\nu(n)}^{-1} x_0 \cdots x_{\nu-1}. \quad (6.32)$$

Hence we obtain by using equations (6.31) and (6.32):

$$\frac{1}{\sqrt{x_0 \cdots x_\nu}} = \mathcal{O}(\sqrt{a_{\nu(n)} n}). \quad (6.33)$$

If we use the trivial estimation ( $x_i x_{i+1} < \frac{1}{2}$ ) we deduce that necessarily  $\forall \epsilon > 0$ :

$$\nu(n) < \frac{2 + \epsilon}{\ln 2} \ln n. \quad (6.34)$$

Thus

- if  $a_n = \mathcal{O}(1)$  then  ${}^1\sigma_n^{[a,b]}(x, \theta) = \mathcal{O}(\sqrt{n})$
- if  $a_n = \mathcal{O}(n^\rho)$  then  ${}^1\sigma_n^{[a,b]}(x, \theta) = \mathcal{O}(\sqrt{n}(\ln n)^{\frac{\rho}{2}})$
- if  $a_n = \mathcal{O}(e^{n^\rho})$ , with  $0 < \rho < \frac{\ln 2}{2}$  then  ${}^1\sigma_n^{[a,b]}(x, \theta) = \mathcal{O}(\sqrt{nn^{\frac{\rho}{\ln 2} + \epsilon}})$ .

Nevertheless let us recall that the equidistribution of  $e^{i\pi x k^2}$ , for any irrational  $x$ , [K] and [KN] give the upper bound of

$${}^1\sigma_n^{[a,b]}(x, \theta) = o(n). \quad (6.35)$$

Now we must use other results of the number theory concerning the computation of continuous fraction expansion of a fraction of the numbers occurring as arguments in our Gaussian sums,  ${}^1\sigma_{(R_j)}^{[\tilde{0}, \tilde{0}]}\left(-\frac{1}{\lambda\beta}, -\frac{1}{\lambda\beta}\theta\right)$ , of the form

$${}^1\sigma_n^{[a,b]}\left(\frac{ux+v}{tx+w}, \theta\right) \leq K ({}^1\sigma_n^{[0,0]}(x, 0)) \quad (6.36)$$

where  $(u, v, t, w) \in \mathbb{N}^4$ .

We will refer to the article of Raney [R] and to the survey of Van der Poorten [VdP]. Let us introduce some notation

$$\begin{aligned} \Psi_k : ]0, 1[ &\longrightarrow \mathbb{N}^{*\mathbb{N}} \\ x &\longmapsto [a_1, a_2, \dots, a_k] \end{aligned} \tag{6.37}$$

such that

$$\Psi_k(x) = \frac{1}{a_1 + \frac{1}{a_2 + \frac{1}{\dots + \frac{1}{a_k}}}}. \tag{6.38}$$

Denote by  $\Psi_\infty$  the map which associates with any real number its continuous fraction.  $\Psi_\infty$  is a homeomorphism from  $]0, 1[$  to  $\mathbb{N}^{*\mathbb{N}}$ . Referring to the theorem of convergence for continuous fraction, for any  $x \in ]0, 1[$  we have

$$\Psi_\infty^{-1} \circ \Psi_k(x) \xrightarrow[k \rightarrow \infty]{} x. \tag{6.39}$$

Consider now the matrix representation  $\{R, L\}$  of the expansion in continuous fractions of  $x$

$$R = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix} \quad L = \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}. \tag{6.40}$$

Here  $\forall k \in \mathbb{N}^*$  one gets

$$R^k = \begin{bmatrix} 1 & k \\ 0 & 1 \end{bmatrix} \quad L^k = \begin{bmatrix} 1 & 0 \\ k & 1 \end{bmatrix} \tag{6.41}$$

So  $\Psi_k^{-1}([a_1, \dots, a_k]) \in \mathbb{Q}$  is determined by the formula

$$\Psi_k^{-1}([a_1, \dots, a_k]) = \frac{\sum_{i=1}^2 (R^{a_1} L^{a_2} \dots R^{a_k})_{1,i}}{\sum_{i=1}^2 (R^{a_1} L^{a_2} \dots R^{a_k})_{2,i}}. \tag{6.42}$$

Now we use the main result of the articles quoted above.

The continuous fraction of  $\frac{ax+b}{cx+d}$  is related to that of  $x$  if  $ad - ac \neq 0$ . To this end, let us consider the set of  $2 \times 2$  matrices of positive integer entries:

$$\mathcal{E}_n = \left\{ M = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{N}) \mid n = \det M \neq 0, a > c, b < d \right\}. \tag{6.43}$$

The number of such matrices is finite [R]. Denote by  $\mathcal{E}_n = \{A_1, \dots, A_N\}$ . The commutation relations obey some restrictions:  $\forall i = 1, \dots, N, \exists i', i'' = 1, \dots, N$  such that

$$A_i R^{k_R^1} L^{k_L^1} = L^{k_L^1} R^{k_R^1} A_{i'}, \quad A_i L^{k_L^2} R^{k_R^2} = R^{k_R^2} L^{k_L^2} A_{i''} \tag{6.44}$$

with  $(k_R^1, k_L^1)$  positive exponents such that  $1 \leq k_R^1 + k_L^1 \leq n$  and so on. The algorithm which allows us to compute, for a given  $n$ , the matrices on the left-hand side is given below. We note that these commutation relations are invariant under the transposition

$$A_i R^{k_R^1} L^{k_L^1} = L^{k_L^1} R^{k_R^1} A_{i'}, \quad {}^t A_{i'} R^{k_R^1} L^{k_L^1} = L^{k_L^1} R^{k_R^1} {}^t A_i. \tag{6.45}$$

Note that for any  $n \in \mathbb{N}^*$

$$\begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} R = R^n \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} \quad \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix} L = L^n \begin{pmatrix} n & 0 \\ 0 & 1 \end{pmatrix}. \tag{6.46}$$

This provides an algorithm to compute the expansion into continued fraction of the values of the function  $x \mapsto \frac{ax+b}{cx+d}$ .

6.2. Illustration of the algorithm

We now shall illustrate this method for  $n = 2$ . The set  $\mathcal{E}_2$  is made up of two matrices which represent the multiplication and the division by 2:

$$\mathcal{E}_2 = \{A, A'\} \tag{6.47}$$

$$A = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix} A' = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix}. \tag{6.48}$$

The rules of commutation are compiled in the following tabulation:

	$A$	$A'$	
$A$	$R : R^2$	$LR : RL$	
	$L^2 : L$		
$A'$	$RL : LR$	$R^2 : R$	
	$L : L^2$		

The table is to be read following the example of the first entry:  $AR = R^2A$ . These relations are equivalent to the following distinct cases:

$$\begin{aligned} x = [0, 2a, b, \dots] &\Rightarrow 2x = [0, a, 2b, \dots] \\ x = [0, 2a + 1, b, \dots] &\Rightarrow 2x = [0, a, 1, 1, (b - 1)/2, \dots]. \end{aligned} \tag{6.50}$$

The entire determination of  $2x$  from  $x$  is done by induction with respect to those rules. For example, if  $x = [7, 2, 5, \dots]$  then

$$2x = [3, 1, 1, 0, 1, 1, 2, \dots] = [3, 1, 2, 1, 2, \dots]. \tag{6.51}$$

Let us introduce some notation:  $[x_0, x_1, \dots]$  with  $0 < x_i < 1, \forall i \in \mathbb{N}$ , the sequence of the rest deduced by the decomposition into continuous fractions of  $x = x_0$ . We shall denote the resulting sequence of the rest of  $2x$  by  $2x = x'_0 = [x'_0, x'_1, \dots]$ . Thus we have, by definition, the relation

$$x_i = -b_{i+1} + \frac{1}{x_{i+1}} \tag{6.52}$$

where  $x = [0; b_1, b_2, \dots]$ . As formula (equation (6.50)) shows, the processes of commutation (resulting of the multiplication by 2 generates a lag length between the two sequences ( $x = [0; b_1, b_2, \dots]$  and  $2x = [0; b'_1, b'_2, \dots]$ ). We shall define a function (depending on the decomposition of  $x$ ) which measures the length of  $2x$  with respect to  $x$ :

$$\begin{aligned} k'_x: \mathbb{N} &\longrightarrow \mathbb{N} \\ k &\longmapsto k'_x(k) \end{aligned} \tag{6.53}$$

such that

- if 2 divides  $b_1$  (in what follows denoted as:  $2 \mid b_1$ ), then  $k'_x(k_1) = k_1$  and  $k'_x(k_2) = k_2$
- if 2 does not divide  $b_1$  (in what follows denoted as  $2 \nmid b_1$ ) then  $k'_x(k_1) = k_1 + 3$  and
  - (a) if  $2 \nmid b_2$  then  $k'_x(k_2) = k'_x(k_1) + 1$
  - (b) if  $2 \mid b_2$  then  $k'_x(k_2) = k'_x(k_1) + 3$

and so on. We recall the trivial reduction which occurs when one of the components of the decomposition is zero, we shall refer to the example (equation (6.51)). Note for this  $x$ ,  $k'_x(3) = 5$ .

**Lemma 15.**  $\forall p \in \mathbb{N}$ , let  $[x_0, x_1, \dots], [x'_0, x'_1, \dots]$  be the rests of the expansion of  $x$  and  $x' = px$  respectively, then  $\forall k \in \mathbb{N} \exists C \in \mathbb{N}$  which divides  $k$  (possibly  $C = 1$ ) such that

$$x'_1 \dots x'_{k'_x(k)} = Cx_1 \dots x_k \tag{6.54}$$

where  $[x_0, x_1, \dots], [x'_0, x'_1, \dots]$  are respectively the rest sequence of  $x$  and  $px$ .

**Proof.** We shall reduce the proof to the simplest case of  $p = 2$ . The other case is quite the same but it turns out to be more difficult because of an involved algebra. We prove the lemma distinguishing all the possibilities:

- if  $2 \mid b_1$  then  $b'_1 = \frac{b_1}{2}$ . Compute now  $x'_0x'_1 = x'_0(-b'_1 + \frac{1}{x'_0}) = 1 - b_1x_0 = x_0x_1$ . Consequently  $x'_1 = \frac{x_1}{2}$ . Since  $2 \mid b_1$  then  $b'_2 = 2b_2$  and  $x'_1x'_2 = 1 - b'_2x'_1 = 1 - b_2x_1 = x_1x_2$  and  $x'_2 = 2x_1$ .  
At this level we have two possibilities either  $2 \mid b_3$  and the procedure repeats itself identically or  $2 \nmid b_3$ , in this case we shall refer to the following:
- if  $2 \nmid b_1$  then  $b'_1 = \frac{b_1-1}{2}$  and  $b'_2 = b'_3 = 1$ . Hence let us calculate  $x'_0x'_1x'_2x'_3$ :  $x'_0x'_1x'_2x'_3 = -(2b'_1 + 1)x'_0 + 2 = -2b_1x_0 + 2 = 2(1 - b_1x_0) = 2x_1x_0$  so  $x_1 = x'_1x'_2x'_3$ . We establish another relation  $x'_0x'_1x'_2 = (1 + b'_1)x'_0 + 1$  then  $x'_1x'_2 = 1 + b'_1 - \frac{1}{x'_0} = \frac{1-x_1}{2}$  ie  $x_1 + 2x'_1x'_2 = 1$  and we deduce that  $(1 - x_1)x'_3 = 2x_1$
- if  $2 \nmid b_2$  then  $b'_4 = \frac{b_2-1}{2}$  and  $b'_5 = 2b_3$ . Calculating  $x'_3x'_4x'_5 = x'_3(1 + b'_5b'_4) - b'_5$ , so  $x'_4x'_5 = 1 + b_2b_3 - \frac{b_3}{x_1} = x_2x_3$ . We also prove that  $-b'_5x'_4 + 1 = x_2x_3 = -b_3x_2 + 1$  so  $x'_4 = \frac{x_2}{2}$  and  $x'_5 = 2x_3$
- if  $2 \mid b_2$  then  $b'_4 = \frac{b_2-2}{2}$  end  $b'_5 = b'_6 = 1$ .  $x'_3x'_4x'_5x'_6 = 2 - (b_2 - 1)x'_3$ . So using  $2x_1 = x'_3(1 - x_1)$  we get  $x'_4x'_5x'_6 = x_2$ . we also get  $2x'_4x'_5 + x_2 = 1$ . So let us summarize these results in the following tables:

$$\begin{array}{cccccccc}
 x = & 2b_1 & b_2 & 2b_3 & & & & \\
 x_0 & x_1 & x_2 & x_3 & & & & \\
 2x = & b_1 & 2b_2 & b_3 & & & & \\
 x'_0 = 2x_0 & x'_1 = \frac{x_1}{2} & x'_2 = 2x_2 & x'_3 = \frac{x_3}{2} & & & & \\
 x = & 2b_1 & b_2 & 2b_3 + 1 & & & & \\
 x_0 & x_1 & x_2 & x_3 & & & & \\
 2x = & b_1 & 2b_2 & b_3 & 1 & 1 & & \\
 & & & & x'_3 & & & \\
 x'_0 = 2x_0 & x'_1 = \frac{x_1}{2} & x'_2 = 2x_2 & x_3 = x'_3x'_4x'_5 & x'_4 & x'_5 & & \\
 & & & x'_3x'_4 + 2x_3 = 1 & & & & 
 \end{array} \tag{6.55}$$

and we also have

$$\begin{array}{cccccccc}
 x = & 2b_1 + 1 & & 2b_2 + 1 & b_3 & & & \\
 x_0 & x_1 & & x_2 & x_3 & & & \\
 2x = & b_1 & 1 & 1 & b_2 & 2b_3 & & \\
 & x'_1 & & & & & & \\
 x'_0 = 2x_0 & x_1 = x'_1x'_2x'_3 & x'_2 & x'_3 & \frac{x_2}{2} & 2x_3 & & \\
 & x'_1x'_2 + 2x_1 = 1 & & & & & & \\
 x = & 2b_1 + 1 & & 2b_2 + 2 & & & & \\
 x_0 & x_1 & & x_2 & & & & \\
 2x = & b_1 & 1 & 1 & b_2 & 1 & 1 & \\
 & x'_1 & & & x'_4 & & & \\
 x'_0 = 2x_0 & x_1 = x'_1x'_2x'_3 & x'_2 & x'_3 & x_2 = x'_4x'_5x'_6 & x'_5 & x'_6 & \\
 & x'_1x'_2 + 2x_1 = 1 & & & x'_4x'_5 + 2x_2 = 1 & & & 
 \end{array} \tag{6.56}$$

Thus we have:  $\forall k \in \mathbb{N} x_1 \dots x_k = x'_1 \dots x'_{k'(k)}$  or  $2x_1 \dots x_k = x'_1 \dots x'_{k'(k)}$ . This complete the proof of the lemma.  $\square$

More generically, let  $A \in \mathcal{E}_n$  goes through the string  $R^{a_1} L^{a_2} R^{a_3} \dots$ , commuting with respect to the two matrices  $\{R, L\}$ :  $[AR^{a_1} L^{a_2} R^{a_3} \dots R^{a_k}] \mapsto [R^{a'_1} L^{a'_2} R^{a'_3} \dots R^{a'_{k'}} A']$  with  $A' \in \mathcal{E}_n$ . This procedure gives an algorithm for the computation of

$$\Psi_{k'(k)} \left( \frac{ax + b}{cx + d} \right) = [a'_1, a'_2, a'_3, \dots, a'_{k'(k)}] \tag{6.57}$$

which only depends on  $[a_1, a_2, \dots, a_k]$ . Now using the above lemma in the formula (equation (6.28)), we obtain the theorem:

**Theorem 6.1.**  $\forall (u, v, t, w) \in \mathbb{N}^4, \forall x \in \mathbb{R} \setminus \mathbb{Q}, \forall \theta \in [0, 1[$  and  $\forall (a, b) \in \{0, 1\}$ , then

$${}^1\sigma_n^{[a,b]} \left( \frac{ux + v}{tx + w}, \theta \right) = \mathcal{O} \left( {}^1\sigma_n^{[0,0]}(x, 0) \right). \tag{6.58}$$

By applying this theorem to the above expression one can deduce that

$$\xi_n(x, \beta; P_2, \dots, P_{2q}) = \sqrt{\frac{\sum P_i}{P_2 \dots P_{2q}}} \sum_{k_1=0}^{n-1} \sum_{k_{2q}=0}^{n-1} e^{2i\pi x(k_{2q}-k_1)(\sum P_i)} \mathcal{O} \left( {}^1\sigma_n^{[a,b]}(\beta, 0) \right)^{2q-2}. \tag{6.59}$$

**Remark 16.** The estimation depends on  $(u, v, t, w)$ . There remains an essential question: does there exist a constant  $K$  independent of  $(u, v, t, w) \in \mathbb{N}^4$  such that

$${}^1\sigma_n^{[a,b]} \left( \frac{ux + v}{tx + w}, \theta \right) \leq K \left( {}^1\sigma_n^{[0,0]}(x, 0) \right)? \tag{6.60}$$

This result is not trivial to prove. In fact, if this assertion were true, the characteristic function would be analytical in a neighbourhood of zero. Consequently, this work only gives a partial answer.

**Remark 17.** We observe that this result is an optimal estimation [HL, p 225, theorem 2.221] when applied to the matrix case: for any  $\Lambda$  is an integer invertible matrix, and for any  $\beta \in \mathbb{R} \setminus \mathbb{Q}$ , we have

$${}^d\sigma_{A_n}^{[\vec{a}, \vec{b}]}(\beta \Lambda, \vec{\theta}) = \mathcal{O} \left( {}^1\sigma_n^{[0,0]}(\beta, 0) \right)^d. \tag{6.61}$$

It cannot be replaced by a better estimate.

Now we can compute the mean value of  $S_n^{2q} \left( \frac{\pi\beta}{4} \right)$  (see equation (3.5)):

$$\begin{aligned} E \left[ S_n^{2q} \left( \frac{\pi\beta}{4} \right) \right] &= (-1)^q \left( \frac{4}{\pi} \right)^q \sum_{P_i \in \mathbb{Z}} \frac{1}{\prod P_i (\sum P_i)} \int_0^1 \xi_n(x, \beta; P_2, \dots, P_{2q}) dx \\ &= (-1)^q \left( \frac{4}{\pi} \right)^q \sum_{P_i \in \mathbb{Z}} \frac{\mathcal{O} \left( {}^1\sigma_n^{[0,0]}(\beta, 0) \right)^{2(q-1)}}{(\prod P_i)^{\frac{3}{2}} \sqrt{\sum P_i}} \sum_{k_1=0}^{n-1} \sum_{k_{2q}=0}^{n-1} \int_0^1 e^{2i\pi x(k_{2q}-k_1) \sum P_i} dx \end{aligned} \tag{6.62}$$

where  $\prod$  stands for  $\prod_{i=2}^{2q}$ . Performing the integral in  $x$  yields the condition  $k_1 = k_{2q}$ , we end up with

$$\begin{aligned} E \left[ S_n^{2q} \left( \frac{\pi\beta}{4} \right) \right] &= n \sum_{P_i \in \mathbb{Z}} \frac{\mathcal{O} \left( {}^1\sigma_n^{[0,0]}(\beta, 0) \right)^{2(q-1)}}{(\prod P_i)^{\frac{3}{2}} \sqrt{\sum P_i}} \\ &= n \cdot c_q \cdot \mathcal{O} \left( {}^1\sigma_n^{[0,0]}(\beta, 0) \right)^{2(q-1)}. \end{aligned} \tag{6.63}$$

$\mathcal{O}({}^1\sigma_n^{[0,0]}(x, 0))^{2(q-1)}$  depends on  $n$  but also on  $(P_2, \dots, P_{2(q-1)})$ . Moreover  $c_q$  is finite because the moments are perfectly defined (the integration is done over a compact set  $[0, 2\pi] \times [0, 2\pi]$ ). So

$$E \left[ S_n^{2q} \left( \frac{\pi\beta}{4} \right) \right] = n \cdot \mathcal{O}({}^1\sigma_n^{[0,0]}(\beta, 0)^{2(q-1)}). \tag{6.64}$$

**Remark 18.** For  $q = 1$ , we recover the exact result  $E[S_n^{2q}(\frac{\pi\beta}{4})] = n$ , which is independent of the choice of  $\beta$ .

**Theorem 6.2.** *The unique sequence  $f_n$  (up to an equivalence as  $n \rightarrow \infty$ ) in order that*

$$E \left[ \frac{S_n(\frac{\pi\beta}{4})}{f_n} \right]^{2q} = \mathcal{O}(1) \tag{6.65}$$

with respect to  $n$  for all  $q$  is  $f_n = \sqrt{n}$ . This estimate is true for all  $\beta$  such that  ${}^1\sigma_n^{[0,0]}(\beta, 0) = \mathcal{O}(\sqrt{n})$ .

**Proof.** We must choose a  $\beta$  having a bounded continuous fractions representation to have  $f_n$  independent of  $q$ . The proof results straightforwardly from the remark 14 that is

$$\frac{f_n^2}{n^q} \left( \frac{{}^1\sigma_n^{[0,0]}(\frac{\pi\beta}{4}, 0)}{f_n} \right)^{2q} = \mathcal{O}(1). \tag{6.66}$$

□

**Remark 19.** Note that if  $\beta \in \mathbb{Q}$  then [HL]  ${}^1\sigma_n^{[0,0]}(\beta, 0) = \mathcal{O}(n)$  and we recover the result established earlier, i.e.

$$E \left[ S_n^{2q} \left( \frac{\pi\beta}{4} \right) \right] = \mathcal{O}(n^{2q-1}) \tag{6.67}$$

which means that there is no normalization of  $S_n$  for which the moments neither diverge nor be nonzero.

Examples of numbers  $\beta$  which admit an expansion as an infinite and bounded continued fractions are the quadratic irrationals. They ultimately have periodic continued fraction expansion. This class contains all the square roots of products of pairwise distinct prime integers, and of course we have, for those numbers,  $\forall q \in \mathbb{N}$

$$E \left[ S_n^{2q} \left( \frac{\pi\beta}{4} \right) \right] = \mathcal{O}(n^q). \tag{6.68}$$

Hence there exists a normalization  $\sqrt{n}$  of  $S_n$  for which all moments remain bounded with respect to  $n$  and are not zero.

**Remark 6.3.** Any attempt to obtain information about the convergence in the distribution of  $S_n/f_n$  by using numerical computations cannot be correct since behaviours are radically different according to the nature of the number  $\beta$  and the rational numbers are dense in  $\mathbb{R}$ . Finally all of these calculations may be applied in the same way to any periodic function instead of the signum function  $\chi(y)$ .

As to the problem of the convergence in the distribution of  $S_n/\sqrt{n}$ , we shall make some comments in the next section.



## 7. Conclusions

In this approach to determine the limiting distribution law of  $\frac{S_n}{\sqrt{n}}$  we have sought to compute the asymptotic behaviour of the moments of  $\frac{S_n}{\sqrt{n}}$ . If the parameter  $\beta = \frac{\alpha}{4\pi}$  admits an expansion in bounded continued fraction, the behaviour obtained for the expectation values of moments  $E\left(\frac{S_n^{2k}}{n^k}\right) = \mathcal{O}(1) \leq A_k$  may lead to the convergence of the series  $\sum_{k=0}^{\infty} \frac{(it)^{2k}}{2k!} A_k$ , around  $t = 0$ . In that case, the sequence  $\Phi_n(t)$  converges towards  $\Phi(t)$ , which is analytic near the origin. This implies the existence of a limiting distribution with finite moments. The estimation of the speed of the increase of the  $A_k$  seems difficult and is still an open problem. It is thus difficult to have an idea of the limiting distribution without such an estimation. Although the procedure does not imply the convergence in distribution for  $\frac{S_n}{\sqrt{n}}$ , it shows that this normalization leading to a bounded second moment fails to lead to bounded moments of higher orders when  $\beta$  is an irrational having no expansion in bounded continued fractions.

## Acknowledgment

It is a pleasure to thank Martine Queffelec for exchange of information about works on Gaussian sums and continued fraction decomposition.

## References

- [AR] Armitage V and Rogers A 2000 Gauss sums and quantum mechanics *J. Phys. A: Math. Gen.* **33** 5993–6002
- [Bi] Billingsley P 1978 *Convergence of Probability Measures* (New York: Wiley)
- [BCS] Bunimovich L A, Chernov N I and Bunimovich Ya 1991 Statistical properties of two-dimensional hyperbolic billiards *Russ. Math. Survey* **46** 47–106
- [BS] Bunimovich L A and Sinai Ya 1980/81 Markov partitions for dispersed billiards *Commun. Math. Phys.* **78** 247–80
- [CC] Crawford J D and Cary J R 1983 Decay of correlations in a chaotic measure preserving transformation *Physica D* **6** 23–232
- [CH1] Courbage M and Hamdan D 1997 Decay of correlation and mixing properties in a dynamical system with zero entropy *Ergod. Theor. Dynam. Syst.* **17** 87–103
- [CH2] Courbage M and Hamdan D 1994 Chapman–Kolmogorov equation for non-markovian shift invariant measure *Ann. Prob.* **22** 1662–77
- [CH3] Courbage M and Hamdan D 1995 Unpredictability in some nonchaotic dynamical systems *Phys. Rev. Lett.* **74** 5166–9
- [CFS] Cornfel I P, Fomin S V and Sinai Ya G 1981 *Ergodic Theory* (New York: Springer)
- [CYo] Chernov N I and Young L S 1998 Statistical properties of dynamical systems with some hyperbolicity *Ann. Math.* **147** 585–650
- [DM] Dekking F M and Mendés France M 1981 Uniform distribution modulo one: a geometrical viewpoint *J. Reine Angew. Math.* **329** 143–53
- [Go] Gordin M I 1969 The central limit theorem for stationary processes (Russian) *Dokl. Akad. Nauk SSSR* **188** 739–41
- [HL] Hardy G H and Littlewood J E 1914 Problems of diophantine approximation II *Acta Mathematica* **37** 193–238
- [J] Janson S 1988 Some pairwise independent sequences for which the central limit theorem fails *Stochastics* **23** 439–48
- [K] Korobov N M and Niederreiter H 1992 Exponential sums and their applications *Mathematics and Its Applications* (Boston, MA: Kluwer)
- [KN] Kuipers L and Niederreiter H 1994 *Uniform Distribution of Sequences* (New York: Wiley)
- [L] Lindelöf E 1989 Le calcul des résidus et ses applications à la théorie des fonctions (Reprinted from the 1905 original by Les Grands Classiques Gauthier-Villars. editions Jacques Gabay, Sceaux)
- [Li] Liverani C 1996 Central limit for deterministic systems *Int. Conf. on Dynamical Systems (Montevideo, 1995)* (*Pitman Res. Notes Math. Ser.* vol 362) (Harlow: Longman) pp 56–75

- [M] Munford D 1983 *Tata Lectures on Theta Functions* vol 1 (Boston, MA: Birkhauser)
- [MMP] Mackay R S, Meiss J D and Percival I C 1984 *Transport in Hamiltonian Systems Physica D* **13** 55–81
- [MOS] Magnus W, Oberhettinger F and Soni R P 1966 *Formulae and Theorems for Special Functions of Mathematical Physics* 3rd edn (New York: Springer)
- [R] Raney N 1973 On Continued Fractions and Finite Automata *Math. Ann.* **206** 265–83
- [SLB] Seshadri S, Lahshamibala S and Balakrishnan V 2000 Control of wave packet revivals using geometric phases *J. Stat. Phys.* **101** 213–23
- [VdP] Van der Porteen A J 1985 *Diophantine Analysis: Roceeding of The Number Theory Section of The 1985 Australian Mathematical Society Convention* (New York: Cambridge University Press)
- [Z] Zalcwasser Z 1937 Sur les polynômes associés aux fonctions modulaires Theta *Stud. Math.* **7** 16–35
- [Za] Zaslavsky G M 1999 Chaotic dynamics and the origin of statistical laws *Phys. Today* August